# Efficiency of market-based resource allocation among many participants

Jia Yuan Yu and Shie Mannor

***Abstract*— Market mechanisms have been suggested in the last few years as a tool for allocating shared networks resources among several competing users. In this paper, we consider the efficiency loss of such mechanisms in the presence of a large number of users. We model the user interactions as a game with a heterogeneous population of players characterized by random utility functions. If the utility functions are bounded, then the non-cooperative equilibrium are nearly as efficient as the social optimum with high probability when the number of users is large. This efficiency result holds for a single link with a fixed or an increasing capacity. Using a standard probabilistic analysis, we show that the efficiency loss incurred by the market mechanism decreases almost exponentially in the number of users. If, however, the utility functions are not bounded, then the loss of efficiency does not converge to zero. We also provide results for networks by sampling the users at random based on their paths.**

***Index Terms*— Game theory, convex optimization, games with many players, extreme value theory, network resource allocation.**

## I. INTRODUCTION

A fundamental challenge in the operation of large-scale broadband communication networks is how to utilize the network resource (*e.g.*, bandwidth) efficiently. Heterogeneous users with a range of demand in terms of transmission rate are sharing the same network. This gives rise to the need to capture each user's perceived utility from its allocation of the network resource. Furthermore, as a result of the large and sprawling nature of modern networks, it is difficult to manage the resource in a centralized fashion. An alternative approach is to allow the users to compete for the network resource, as a commodity, through a market mechanism.

We compare the efficiency of two paradigms: the first is a centralized resource allocation mechanism where an agent divides the network's resource such that the aggregate utility of all users is maximized, the second is a decentralized market mechanism where users receive amounts of resource proportional to their bids. In terms of the aggregate utility of all the network's users, the centralized allocation mechanism achieves an optimal solution. Kelly [Kel97] shows that the market mechanism also admits a competitive equilibrium with an optimal solution, provided that the users do not anticipate the effect of their bids on the market price. However, from a game theoretic point of view, it is natural to assume that users do anticipate the effect of their bids. When the users are price-anticipators, the market mechanism admits nevertheless a Nash equilibrium for which the efficiency loss is at most $25\%$ for inelastic supply [JT04], and at most approximately $34\%$ for elastic supply (where supply is dictated by demand) [JMT05]. Building upon these encouraging results, we show that, under some standard assumptions, the loss of efficiency actually tends to 0 when the number of potential users is large.

Comparing socially optimal and equilibrium outcomes in the context of networks has attracted great interest since the work of Koutsoupias and Papadimitriou [KP99]. For the routing problem, Roughgarden and Tardos [RT02] show that the total latency experienced by selfish users is at most $4/3$ times the minimum total latency. Moreover, empirical study of selfish routing in realistic environments (*e.g.*, the Internet) already suggests that the loss of efficiency is often much less than the worst-case bound [QYZS03]. Friedman [Fri04] shows that the performance degradation due to selfish objectives is generally small.

In this paper, we study the efficiency loss due to lack of cooperation among users, which can also be attributed to price-anticipating behavior in a market environment. Hence, we assume that the central agent has perfect knowledge of the true utilities, and that every user has knowledge of the sum of the other users' bids and acts optimally with regards to its own interest. Loss of efficiency can also be the result of adversarial, non-optimal, or collusive behavior among the users; these are not studied here. We will focus on asymptotic results for large and heterogeneous populations of users. Our analysis employ standard probabilistic techniques from machine learning.

The paper is organized as follows. The market-based resource allocation problem is described in Section II. We show that, under some standard conditions, the loss of efficiency tends to 0 almost surely as the number of users tends to infinity. We first prove this result for a single link (Section III), then provide partial results for general networks (Section V). For the single-link case, the result holds whether the link capacity is fixed (Section III-A) or scales according to the number of users (Section III-B). Moreover, a probabilistic analysis shows that the efficiency loss vanishes at a rate that is approximately exponential (Section III-A). However, under different assumptions, we use the theory of extreme values to show that the loss of efficiency does not tend to 0 (Section IV). The positive and negative results are interspersed with illustrative simulations. In Section VI, we discuss our work in perspective with related works in economics.

---

The authors are with the Electrical and Computer Engineering Department of McGill University. J. Y. Yu was partially supported by NSERC graduate scholarship. S. Mannor was partially supported by NSERC and the Canada Research Chairs program.

The authors would like to thank the reviewers.

A preliminary version of this paper [YM06] was presented at INFOCOM 2006.

## II. BACKGROUND

In this section, we present the essential model, definitions and assumptions used in the subsequent development. For now, we restrict our attention to a single link. In Section II-A, we characterize the users and their utility functions. In Section II-B, we recall the market-based resource allocation mechanism based on [Kel97] and describe the resulting user interactions from a game theoretic perspective. In Section II-C, we specify the probabilistic setup for random utility functions.

### A. Utility functions

Consider a set of users, labeled $1, \ldots, n$, that share an amount $c$ of a given network resource. Henceforth, we will refer to this resource as the *commodity*, whether it is traffic, bandwidth, time-slots, etc. User $i$ is characterized by the utility function $u_i : \mathbb{R}_+ \to \mathbb{R}_+$. We will make the following assumption concerning the utility functions.

*Assumption 2.1:* The utility functions $\{u_i\}$:

i. are concave and continuously differentiable,
ii. are strictly increasing, with $u_i(0) = 0$,
iii. have finite slope everywhere, *i.e.*, for all $z > 0$, we have $u_i'(z) < \infty$.

Observe that this assumption allows linear utility functions (*cf.* [Kel97] where a strict concavity assumption is made). Traffic that leads to such utility functions is referred to as *elastic* traffic [She95].

In general, valid utility functions need not have a closed-form expression. We could dispense with the continuous derivative assumption to the detriment of ease of analysis. The strictly increasing condition ensures that the amount of resource $c$ is entirely allocated. The condition $u_i(0) = 0$ is reflects the notion that no commodity typically implies no benefit. For brevity, we will write $u_i'(0)$ for the *initial slope* of user $i$, *i.e.*, for $\lim_{\epsilon \downarrow 0} u_i'(\epsilon)$. Note that Assumption 2.1 does not allow constantly zero functions.

### B. Optimal and equilibrium outcomes

We now specify the assumptions of the market mechanism model; for more details, the reader is referred to [Kel97]. For the ease of exposition, the number of users and their utility functions are assumed fixed (and non-random) in this subsection.

Let $x_i$ be the amount of commodity allocated to user $i$. An allocation $(x_1, \ldots, x_n)$ achieves social optimality if it is a solution to the optimization problem

**SYSTEM**

$$\begin{aligned} \max_{x_1,\ldots,x_n} \quad & \sum_{i=1}^n u_i(x_i) \\ \text{subject to} \quad & \sum_{i=1}^n x_i \leq c, \\ & x_i \geq 0, \quad i = 1, \ldots, n. \end{aligned} \tag{1}$$

We call the objective function of the problem SYSTEM the *aggregate utility*. A central agent with knowledge of the utility functions can implement this allocation mechanism.

We will consider the bidding mechanism, introduced by Kelly [Kel97], where the users bid for the resource and receive proportional shares. A general version of the mechanism is described by Shubik [Shu73] and Shapley [Sha76]. Specifically,

1) every user $i$ submits a bid $b_i \in \mathbb{R}_+$,
2) if $\sum_{i=1}^n b_i > 0$, then the price for the commodity is set to
$$\lambda = \frac{\sum_{i=1}^n b_i}{c},$$
3) user $i$ receives an amount $x_i = b_i/\lambda$ of the commodity.

In this setting, users can act as *price-takers* or *price-anticipators*. Price-taking users do not anticipate the effect of their bid on the price. When $\sum_{i=1}^n b_i > 0$, every price-taking user $i$ maximizes its surplus

$$u_i\left(\frac{b_i}{\lambda}\right) - b_i.$$

In contrast, every price-anticipating user $i$ maximizes the following surplus when $\sum_{i=1}^n b_i > 0$:

$$u_i\left(\frac{b_i}{b_i + \sum_{j \neq i} b_j} c\right) - b_i.$$

Kelly [Kel97] shows that, when the users behave as price-takers[1], there exists a competitive equilibrium, whose corresponding allocation is an optimal solution to SYSTEM. This effectively establishes the equivalence between two interpretations of social optimum: price-taking bidding equilibrium and centralized allocation. Hajek and Gopalakrishnan [HG04] showed the existence of a unique Nash equilibrium under Assumption 2.1 when the users behave as price-anticipators. This Nash equilibrium is the solution to an optimization problem similar to SYSTEM, with modified utility functions.

*Theorem 2.1 (Hajek and Gopalakrishnan, [HG04]):* Assume that there are $n \geq 2$ users, each with a concave, strictly increasing, and continuously differentiable utility function. Then there exists a unique Nash equilibrium, and the resource allocation at the Nash equilibrium is the solution to the following optimization problem:

**GAME**

$$\begin{aligned} \max_{y_1,\ldots,y_n} \quad & \sum_{i=1}^n \left[\left(1 - \frac{y_i}{c}\right) u_i(y_i) + \frac{1}{c}\int_0^{y_i} u_i(z)\,\mathrm{d}z\right] \\ \text{subject to} \quad & \sum_{i=1}^n y_i \leq c, \\ & y_i \geq 0, \quad i = 1, \ldots, n. \end{aligned} \tag{2}$$

A Nash equilibrium in this setting is a set of bids such that no user can increase his utility by unilaterally changing his bid. Let the allocation $y_1, \ldots, y_n$ arise from bids at a Nash equilibrium between price-anticipating users. We adopt the following notion of *price of anarchy* [JT04], drawn from the

---

[1] Kelly's result is under the assumption that the utility functions are *strictly* concave, which is slightly stronger than Assumption 2.1.

analogy between Nash equilibrium and a state where each individual looks exclusively at his own interest:

$$\text{POA} \triangleq \frac{\sum_{i=1}^n u_i(y_i)}{\sum_{i=1}^n u_i(x_i)}.$$

It is also natural to define the more intuitive notion of *loss of efficiency*:

$$\text{LOE} \triangleq 1 - \text{POA}.$$

Intuitively speaking, this is a measure of lost benefit to all users collectively due to lack of cooperation. Johari and Tsitsiklis [JT04] show that the efficiency loss is at most $1/4$ for a single link and for networks.

In the single-link discussion, a “socially optimal outcome,” or simply “optimal outcome,” refers to a solution to SYSTEM, which is equivalent to a competitive equilibrium among price-taking users. By “Nash equilibrium outcome,” or simply “equilibrium outcome,” we mean a solution to GAME, which is equivalent to a Nash equilibrium among price-anticipating users.

*Remark 1:* We do not consider the resource allocation problem from the perspective of the link owner. We assume that the capacity $c$ is chosen in advance.

### C. Modeling a population of random users

In this paper, we consider populations of random users. We will study the asymptotics of the loss of efficiency. By *asymptotic behavior* of efficiency loss, we mean its characteristics when the number of users having random utility functions tends to infinity. As we will see, many of the users may actually receive no commodity at the end of the allocation process, so by “user,” we really mean “potential user.”

We model heterogeneous users by a sequence of random utility functions drawn i.i.d. from a probability space $(\Omega, \mathcal{B}, \Pr)$, where $\Omega$ is a set of utility functions satisfying Assumption 2.1. We will use $U_i$ to denote the random utility functions and $u_i$ for realizations of $U_i$. Let $\nu$ denote the supremum of $u'(0)$ among all functions $u \in \Omega$, *i.e.*, $\nu \triangleq \sup_{u\in\Omega}\{u'(0)\}$. The following is our basic assumption on the set of allowed utility functions $\Omega$.

*Assumption 2.2:* The probability measure $\Pr$ has a positive density on $\Omega$ and the utility functions in $\Omega$ have uniformly bounded slopes at 0, *i.e.*, $\nu < \infty$.

We will show that Assumption 2.2 together with Assumption 2.1 guarantee that, with high probability, no single user has significantly higher marginal utility at 0 than the rest in the many-users limit. In the following, we are interested in what happens in the many-users limit. To this end, we model an increasing sequence of populations of users by the following recursively generated sequence of sets of random utility functions with increasing cardinality:

$$\mathcal{U}^{(1)} = \{U_1\}, \quad \mathcal{U}^{(n)} = \mathcal{U}^{(n-1)} \cup \{U_n\}, \quad \text{for } n \geq 2.$$

Observe that, here and later, when we say that the number of users tends to infinity or write $n \to \infty$, we mean that additional users enter the game while those users already present remain in the game.

A note on the notation is due. We will generally denote random variables with capital letters (*e.g.*, $X_i$, $Y_i$), and their realizations with small letters (*e.g.*, $x_i$, $y_i$). Exceptions to this rule are random variables denoted by Greek letters. In the following, we will allow the utility functions to be random and let $n$ tend to infinity. Consequently, the quantities of interest, such as the price of the commodity $\lambda$, the allocations $X_i$ and $Y_i$, and the efficiency loss LOE, are random quantities that depend on $\mathcal{U}^{(n)}$. For sake of notation, we will simply write the superscript $^{(n)}$ or make this dependence implicit when it is clear from the context.

It would be convenient to consider the randomization mechanism where a scalar random variable multiplies a valid utility function. In that case, we obtain

$$U_i(x) = S_i\, w(x),$$

where $S_i$ are assumed to be random scalars, and $w$ is a deterministic utility function that satisfies Assumption 2.1. We further assume that the sequence $S_1, \ldots, S_n$ is independent and identically distributed according to some discrete or continuous probability distribution. We will call such a randomization mechanism a *scalar modulation* of the utility function $w$. We now present a scalar modulation example that satisfy Assumption 2.2, followed by an example that does not satisfy the assumption.

*Example 2.1 (Modulation satisfying Assumption 2.2):* Suppose that $S_i$ has bounded support and that $w(x) = x$. It is easily verified that $U_i(x) = S_i\, x$ satisfies Assumption 2.2. Same holds for $U_i(x) = S_i \log(1 + x)$.

*Example 2.2 (Modulation violating Assumption 2.2):* The case of $U_i(x) = |S_i|\, x$ with $S_i$ drawn from a Gaussian or exponential distribution violates Assumption 2.2 because $\nu = \infty$.

### D. Optimality conditions

We now recap the set of conditions that characterize socially optimal and Nash equilibrium allocations, as presented in [Kel97] and [JT04]. First, consider the social optimization SYSTEM (1) has the following optimality conditions for a fixed $n$.

- For all $i$ such that $x_i^{(n)} > 0$, we have $u_i'(x_i^{(n)}) = \lambda^{(n)}$. In other words, all active users’ utility functions have the same slope $\lambda^{(n)}$ at their respective allocation levels $x_i^{(n)}$. This condition is guaranteed by the concavity and continuous first derivative of $u_i(x_i^{(n)})$.
- For all $i$ such that $x_i^{(n)} = 0$, we have $u_i'(0) \leq \lambda^{(n)}$.
- $\sum_{i=1}^n x_i^{(n)} = c$. This is true because of the strictly increasing utility assumption.

Note that a user $i$ is active if and only if its initial slope is strictly greater than $\lambda^{(n)}$, *i.e.*, $u_i'(0) > \lambda^{(n)}$, because of the concave utility assumption. Therefore, $\lambda^{(n)}$ represents the shadow price of the commodity.

Similarly, let $y_i^{(n)}$ denote the solution to the Nash equilibrium optimization problem GAME (2). Since the modified utility function from GAME,

$$\tilde{u}_i(y) = \Big(1 - \frac{y}{c}\Big)\, u_i(y) + \frac{1}{c}\int_0^y u_i(z)\,\mathrm{d}z,$$

also satisfies the criteria of Assumption 2.1, we can derive optimality conditions in the same manner.

- For all $i$ such that $y_i^{(n)} > 0$, we have
$$\left(1 - \frac{y_i^{(n)}}{c}\right) u_i'(y_i^{(n)}) = \mu^{(n)},$$
where $\mu^{(n)}$ is the analogue of $\lambda^{(n)}$. This follows from taking the derivative of $\tilde{u}_i(y)$.
- For all $i$ such that $y_i^{(n)} = 0$, we have $u_i'(0) \le \mu^{(n)}$.
- $\sum_{i=1}^n y_i^{(n)} = c$.

Notice the similarity between the two sets of optimality conditions for the socially optimal and equilibrium outcomes, which already hints to the correspondence sought after.

## III. ASYMPTOTIC BEHAVIOR OF EFFICIENCY LOSS FOR A SINGLE LINK

In this section, we establish the convergence of the loss of efficiency to zero under the model and conditions set forth in Section II. Our main theorem says that, with a broad class of utility functions, selfish competition is asymptotically just as efficient as the best possible centrally enforced allocation. We also consider the case where the link capacity scales according to the number of users. Finally, we illustrate the result through simulation with randomly drawn utility functions.

### *A. Rate of convergence*

In this section, we investigate the rate at which the efficiency loss tends to 0 as the number of users increases. We show that the probability that the loss of efficiency is larger than some positive constant decays (almost) exponentially in the number of users. We begin with an example where the decay of this probability is exponential.

Let $\delta > 0$ be fixed. Suppose that we have a set of utility functions $\{w_j\}$ within which is a unique function, say $w_1$, that satisfies, the condition
$$\left(1 - \frac{\delta}{c}\right) w_1'(\delta) > w_j'(0), \quad \text{for all } w_j \neq w_1. \tag{3}$$
If in addition we have at least $\lceil c/\delta \rceil$ users with the utility function $w_1$, then all the capacity will be allocated to users with utility $w_1$ in both the price-taking and price-anticipating outcomes by the optimality conditions. In other words, if the loss of efficiency is greater than 0, then the number of users with utility function $w_1$ must be smaller than $\lceil c/\delta \rceil$. This reasoning translates into:
$$\begin{aligned}&\Pr(\text{LOE}^{(n)} > 0)\\ &\le \sum_{k=0}^{\lfloor c/\delta \rfloor} \binom{n}{k} \Pr(U_i = w_1)^k \big(1 - \Pr(U_i = w_1)\big)^{n-k}.\end{aligned}$$
As shown in the following lemma, this probability approaches zero exponentially fast for large enough $n$.

*Lemma 3.1:* For $m, n \in \mathbb{N}$, $q > 0$, and $n \ge m/q$,
$$\sum_{k=0}^{m} \binom{n}{k} q^k (1-q)^{n-k} \le \exp\left\{-2 \frac{(nq - m)^2}{n}\right\}. \tag{4}$$

*Proof:* The bound follows immediately from the Hoeffding Inequality for Bernoulli random variables [Hoe63]. ■

In the following, we derive the convergence rate of the efficiency loss. We start with the following definition.

*Definition 3.1 (Active users):* The *active users* are the users that receive non-zero allocation. We will employ this term for the socially optimal outcome or the Nash equilibrium, depending on the context.

*Theorem 3.2 (Exponential convergence):* Consider a single-link resource allocation problem with random utility functions satisfying Assumptions 2.1 and 2.2. For a fixed $\epsilon \in (0, \nu)$, there exist $\delta > 0$ and $\psi > 0$ such that, for $n \ge \lfloor c/\delta \rfloor / \psi$,
$$\Pr\left(\text{LOE}^{(n)} > \frac{\epsilon}{\nu}\right) \le \exp\left\{-2 \frac{(n\psi - \lfloor c/\delta \rfloor)^2}{n}\right\}.$$

*Proof:* Let $\mu^{(n)}$ denote the shadow price of the commodity in the problem GAME (2) with utility functions $U_1, \ldots, U_n$, and a fixed $n$. We proceed in two steps. In the first step, we bound the value of $\text{LOE}^{(n)}$ in the event that $\mu^{(n)}$ is at least $\nu - \epsilon$. In the second step, we give a necessary condition for this event and establish its high likeliness.

**Step 1**: Bound $\text{LOE}^{(n)}$ in the event that $\mu^{(n)} \ge \nu - \epsilon$.

Recall that, for the original problem, we denote the aggregate utilities of the socially optimal and Nash equilibrium outcomes by $\sum_{i=1}^n U_i(X_i)$ and $\sum_{i=1}^n U_i(Y_i)$ respectively. We can easily verify that
$$\sum_{i=1}^n U_i(X_i) \le \nu c. \tag{5}$$
Observe also that if $\mu^{(n)} \ge \nu - \epsilon$ for a fixed $\epsilon > 0$, then the optimality conditions ensure that
$$\sum_{i=1}^n U_i(Y_i) = \sum_{i \text{ active}} U_i(Y_i) \ge (\nu - \epsilon)c. \tag{6}$$
Combining Inequalities (5) and (6), we obtain the following, in the event that $\mu^{(n)} \ge \nu - \epsilon$,
$$\begin{aligned}\text{LOE}^{(n)} &= 1 - \frac{\sum_{i=1}^n U_i(Y_i)}{\sum_{i=1}^n U_i(X_i)}\\ &\le 1 - \frac{(\nu - \epsilon)c}{\nu c} \le \frac{\epsilon}{\nu}.\end{aligned}$$

**Step 2**: Bound the probability of the event $\{\mu^{(n)} \ge \nu - \epsilon\}$.

Consider a realization $u_1, \ldots, u_n$ of the random utility functions $U_1, \ldots, U_n$. By Assumption 2.2, for every $\epsilon_1 > 0$, we have $\Pr(U_i'(0) \ge \nu - \epsilon_1) > 0$. By the continuity of the derivative of $u_i$ (Assumption 2.1), for every $\epsilon_2 > 0$, there exists a $\delta_2 > 0$, such that if $\delta_3 \in (0, \delta_2)$, then $u_i'(\delta_3) > u_i'(0) - \epsilon_2$. Combining these two facts, for $\delta_3 \in (0, \delta_2)$, we have
$$\Pr(U_i'(\delta_3) \ge \nu - \epsilon_1 - \epsilon_2) \ge \Pr(U_i'(0) \ge \nu - \epsilon_1) > 0, \tag{7}$$
For a fixed $\epsilon > 0$, we pick a $\delta_4 > 0$ be such that
$$\begin{aligned}&(1 - \delta_4/c)(\nu - \epsilon_1 - \epsilon_2) \ge \nu - \epsilon, \qquad (8)\\ \text{or} \quad &\delta_4 \le \left(1 - \frac{\nu - \epsilon}{\nu - \epsilon_1 - \epsilon_2}\right) c.\end{aligned}$$

Finally, we let $\delta = \min(\delta_3, \delta_4)$ and $\psi = \Pr\left((1-\delta/c)\, U_i'(\delta) \geq \nu - \epsilon\right)$. Observe that by Equations (8) and (7), we have

$$\begin{aligned}\psi &\geq \Pr\left((1-\delta/c)U_i'(\delta) \geq (1-\delta/c)(\nu - \epsilon_1 - \epsilon_2)\right)\\ &\geq \Pr\left(U_i'(\delta) \geq \nu - \epsilon_1 - \epsilon_2\right) > 0.\end{aligned}$$

Let $\Gamma$ be the set of users whose utility functions satisfy $(1-\delta/c)\ u_i'(\delta) \geq \nu - \epsilon$. Suppose that

$$k \triangleq |\Gamma| \geq \lceil c/\delta \rceil, \tag{9}$$

and that we allocate $c$ only among the users in $\Gamma$. One way to do so is to divide $c$ evenly among the $k$ users in $\Gamma$. In this allocation, all users in $\Gamma$ receive $c/k$ units of resource and have marginal utilities $u_i'(c/k) \geq (1-1/k)^{-1}(\nu-\epsilon) > \nu - \epsilon$. Hence, the Nash equilibrium price of the commodity $\mu^{(n)}$ must be greater than $\nu-\epsilon$. Note that by including the remaining $n-k$ users into the allocation process, the price of the commodity can only increase. Therefore, if there are at least $\lceil c/\delta \rceil$ users with high enough utility, *i.e.*, $(1-\delta/c)\ u_i'(\delta) \geq u - \epsilon$, then $\mu^{(n)} > \nu - \epsilon$. It follows that

$$\begin{aligned}&\Pr(\mu^{(n)} \geq \nu - \epsilon) \geq \sum_{k=\lceil c/\delta \rceil}^{n} \binom{n}{k} \psi^k\ (1-\psi)^{n-k}\,,\\ \Rightarrow\quad &\Pr(\mu^{(n)} < \nu - \epsilon) \leq \sum_{k=0}^{\lfloor c/\delta \rfloor} \binom{n}{k}\ \psi^k\ (1-\psi)^{n-k}.\end{aligned} \tag{10}$$

Finally, the stated result follows from Equation (10) and Lemma 3.1. ■

Note that the value of $\delta$ is a free parameter. Choosing a smaller value for $\delta$ results in a larger factor $\psi$ in the exponent, but the term subtracted (*i.e.*, $\lfloor c/\delta \rfloor$) also increases. Furthermore, the value of $\delta$ also affects the lower bound on $n$ (*i.e.*, $\lfloor c/\delta \rfloor/\psi$).

*Remark 2:* In the derivation of the Theorems 3.2, we are not restricted by the fact that users are added into the system one by one as $n$ increases. The same result holds if we sample a distinct set of $n$ i.i.d. users for each value of $n$, so that $\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(n)}$ are mutually independent.

The following corollary follows from Theorem 3.2 by the Borel-Cantelli Lemma.

*Corollary 3.3 (Almost sure convergence):* Consider a resource allocation game with concave utility functions satisfying Assumption 2.1. Suppose that the utility functions are drawn i.i.d. from a distribution satisfying Assumption 2.2. Then the loss of efficiency tends to 0 almost surely as the number of users tends to infinity.

*Remark 3:* The assumptions of the preceding results can be relaxed. Indeed, this result still holds for sets for utility functions where $\nu = \infty$, provided that $\Pr(U_i'(0) = \infty) > 0$ [YM06].

*Remark 4:* The related question of almost sure convergence of the efficiency loss when the capacity is elastic (*i.e.*, dependent on demand) is studied in [YM06].

### *B. Increasing capacity on a single-link*

In this section, we consider the effect of increasing the link capacity $c$ as the number of users grows. We model the capacity as an increasing function $c : \mathbb{N} \to \mathbb{R}_+$ of the number of users $n$. We make the following assumption to exclude the situation where both $c(n)$ and the allowed utility functions are unbounded.

*Assumption 3.1:* For every $u \in \Omega$, the limit $\overline{u} = \lim_{z\to\infty} u(z)$ exists. If $c(n)$ is not bounded, then the utility functions $u \in \Omega$ are uniformly bounded, *i.e.*, there exist constants $\overline{\beta}$ and $\underline{\beta}$ such that for all $u \in \Omega$, we have $\underline{\beta} \leq \overline{u} \leq \overline{\beta}$. Moreover, for every $\epsilon > 0$, there exists a finite $\zeta > 0$ such that for every $u \in \Omega$ and $z \geq \zeta$, we have $\overline{u} - u(z) < \epsilon$.

*Theorem 3.4 (Increasing capacity):* Consider a single link with an amount of commodity $c(n)$ that is asymptotically sublinear ($c(n) \in o(n)$), linear ($c(n) \in \Theta(n)$), or superlinear ($c(n) \in \omega(n)$). Under Assumptions 2.1, 2.2 and 3.1, the loss of efficiency converges to 0 almost surely as the number of users tends to infinity.

*Proof:* When $c(n)$ is bounded, $n$ eventually exceeds $\lfloor c(n)/\delta \rfloor/\psi$ as $n$ increases. Therefore, Theorem 3.2 and Corollary 3.3 still hold with $c$ replaced by $c(n)$. We consider the cases where $c(n)$ is $o(n)$, $\Theta(n)$, and $\omega(n)$ separately.

**Case 1**: $c(n) \in o(n)$.

First, consider a sublinear capacity function $c(n) \in o(n)$. Observe that the convergence result of Theorem 3.2 does not rely on the fact that $c$ is fixed. Hence, it remains valid when we replace the hard capacity $c$ by a capacity function $c(n) \in o(n)$, since $n \geq \lfloor c(n)/\delta \rfloor/\psi$ for large enough $n$. Theorem 3.2 is not useful however when $c(n)$ is linear or superlinear, since the condition $n \geq \lfloor c(n)/\delta \rfloor/\psi$ no longer holds for large $n$.

**Case 2**: $c(n) \in \omega(n)$.

Consider a superlinear capacity function $c(n)$ and an arbitrary realization $u_1, \ldots, u_n$ of $U_1, \ldots, U_n$. By the pigeonhole principle, there must be a pair of users—say users $i_1$ and $i_2$—who receive at least $c(n)/n$ units of commodity in the socially optimal and Nash equilibrium outcomes, respectively. By Assumptions 2.1 and 3.1, as $n \to \infty$, we have

$$\begin{aligned}\lambda^{(n)} &= u_{i_1}'(x_{i_1}^{(n)}) \leq u_{i_1}'(c(n)/n) \to 0,\\ \text{and}\quad \mu^{(n)} &= \left(1 - \frac{y_{i_2}^{(n)}}{c(n)}\right) u_{i_2}'(y_{i_2}^{(n)})\\ &\leq \left(1 - \frac{c(n)/n}{c(n)}\right) u_{i_2}'(c(n)/n) \to 0.\end{aligned}$$

Let $\mathcal{A}_{\text{SO}}$ and $\mathcal{A}_{\text{NE}}$ denote the sets of active users for the socially optimal and Nash equilibrium solutions, respectively. In turn, the optimality conditions (Section II-D) guarantee that $x_i^{(n)} \to \infty$ and $y_j^{(n)} \to \infty$ as $n \to \infty$ for every user $i \in \mathcal{A}_{\text{SO}}$ and every user $j \in \mathcal{A}_{\text{NE}}$. It follows that for every $\epsilon > 0$ and large enough $n$, we have $\overline{u}_i - u_i(x_i^{(n)}) < \epsilon$ and $\overline{u}_j - u_j(y_j^{(n)}) < \epsilon$. Finally, we obtain that

$$\begin{aligned}\text{POA}^{(n)} &= \frac{\sum_{i=1}^n U_i(Y_i^{(n)})}{\sum_{i=1}^n U_i(X_i^{(n)})} \geq \frac{\sum_{j\in\mathcal{A}_{\text{NE}}}(\overline{u}_j - \epsilon)}{\sum_{i=1}^n \overline{u}_i}\\ &\geq 1 - \frac{\sum_{j\notin\mathcal{A}_{\text{NE}}} \overline{u}_j}{\sum_{i=1}^n \overline{u}_i} - \frac{n\epsilon}{\sum_{i=1}^n \overline{u}_i}\\ &\geq 1 - \frac{|\mathcal{A}_{\text{NE}}^{\text{c}}|\,\overline{\beta}}{n\underline{\beta}} - \frac{\epsilon}{\underline{\beta}},\end{aligned} \tag{11}$$

where $|\mathcal{A}_{\text{NE}}^{\text{c}}|$ is the number of users that are inactive in the equilibrium solution. Let $\delta$ be a constant such that $\delta \geq \mu^{(n)}$ and $N^{(n)}(\delta)$ be the number of users with $U_j'(0) \leq \delta$. Since $N^{(n)}(\delta)$ is the sum of i.i.d. Bernoulli random variables, we obtain, as $n \to \infty$,

$$\frac{|\mathcal{A}_{\text{NE}}^{\text{c}}|}{n} \leq \frac{N^{(n)}(\delta)}{n} \xrightarrow{\text{a.s.}} \Pr\left(U_i'(0) \leq \delta\right),$$

which can be made arbitrarily small by choosing $\delta = 2\mu^{(n)}$ for $n$ large enough.

The second fraction of Inequality (11) can also be made arbitrarily small by the choice of $\epsilon$. It follows that the efficiency loss tends to 0 almost surely.

**Case 3**: $c(n) \in \Theta(n)$.

For a linear capacity function $c(n)$, we adopt a different approach since we do not expect the prices $\lambda^{(n)}$ and $\mu^{(n)}$ to converge to a constant. Consider first a probability space with a finitely many utility functions, $\Omega = \{w_1, \ldots, w_m\}$. Let $u_1, \ldots, u_n$ denote a realization of $U_1, \ldots, U_n$, and let $N_j^{(n)}$ be the number of users whose utility function is $w_j$, and $Y_i^{(n)}$ be the amount of commodity allocated to each such user. First by symmetry of the equilibrium allocation, and then by the strong law of large numbers, we obtain

$$\begin{aligned} & Y_i^{(n)} N_j^{(n)} \leq c(n), \\ \Rightarrow \quad & Y_i^{(n)} \leq \frac{c(n)}{N_j^{(n)}} \xrightarrow{\text{a.s.}} \frac{c'}{\Pr(U_i = w_j)} \quad \text{as } n \to \infty, \end{aligned}$$

where $c' \triangleq \lim_{n\to\infty} c(n)/n$. Combining this result with the conditions for social optimality and Nash equilibrium of Section II-D, we conclude that the optimality conditions coincide in the limit. Hence, likewise, the loss of efficiency tends to 0 almost surely.

To extend this result to a general set of functions $\Omega$, we will show that the set of functions satisfying Assumptions 2.1, 2.2, and 3.1 can be closely approximated by a finite set of functions. Fix $\epsilon > 0$ and let $\zeta > 0$ be a finite constant such that for every $u \in \Omega$ and every $z \geq \zeta$, we have

$$u(z) - u(\zeta) < \epsilon,$$

as guaranteed by the assumptions. We pick a partition of $\Omega$ restricted to $[0, \zeta]$, which is also an $\epsilon$-net[2] [AB99]. Such an $\epsilon$-net exists because $\zeta$ is finite, the members of $\Omega$ are uniformly Lipschitz, and $\Omega$ is uniformly bounded. Let this $\epsilon$-net be denoted by $\Omega_1, \ldots, \Omega_\ell$, where $\ell$ is a function of $\epsilon$. Given the $\epsilon$-net $\Omega_1, \ldots, \Omega_\ell$, we define the following $\ell$ representative functions: $\overline{\omega}_k(z) = \inf_{u \in \Omega_k} u(z)$, for $k = 1, \ldots, \ell$. Since $\overline{\omega}_k$ is the infimum of concave functions, it is also a concave function, but it may not belong to $\Omega$. Moreover, it follows that for every $u \in \Omega_k$ and $z \in \mathbb{R}_+$, we have $0 \leq u(z) - \overline{\omega}_k(z) \leq \epsilon$.

Consider an arbitrary realization $u_1, \ldots, u_n$ of the random utility functions. To each $u_i$ we associate a representative function $\hat{u}_i = \overline{\omega}_k$ if $u_i \in \Omega_k$. By construction and Assumption 3.1, $u_i$ and $\hat{u}_i$ are such that:

$$|u_i(z) - \hat{u}_i(z)| \leq \epsilon \quad \text{for every } z \geq 0. \tag{12}$$

Let $(\hat{x}_i)$ and $(\hat{y}_i)$ denote the optimal and equilibrium allocations, respectively, for users with the utility functions $\hat{u}_1, \ldots, \hat{u}_n$ instead of $u_1, \ldots, u_n$. By definition, we have $\sum_{i=1}^n \hat{u}_i(\hat{x}_i) \geq \sum_{i=1}^n \hat{u}_i(x_i)$. Combined with Equation (12), we obtain

$$\sum_{i=1}^n u_i(x_i) \leq \sum_{i=1}^n (\hat{u}_i(x_i) + \epsilon) \leq \sum_{i=1}^n \hat{u}_i(\hat{x}_i) + \epsilon n. \tag{13}$$

Our next goal is to bound $\sum_{i=1}^n \hat{u}_i(\hat{y}_i)$ from below by $\sum_{i=1}^n u_i(y_i) - \Delta(\epsilon)n$, where $\Delta(\epsilon)$ satisfies that $\lim_{\epsilon\downarrow 0} \Delta(\epsilon) = 0$. Let us consider four optimization problems. The first problem is GAME (2) as applied to the sampled utility functions $u_1, \ldots, u_n$. The second problem is a modified version of GAME where users belonging to the same partition $\Omega_k$ are constrained to have the same allocation. In the third optimization problem, users in the same partition $\Omega_k$ are constrained to have the same allocation, and assigned the same (infimal) representative utility function $\overline{\omega}_k$. The fourth optimization problem is GAME as applied to the representatives $\hat{u}_1, \ldots, \hat{u}_n$. We show that, for large enough $n$, the aggregate utility between solutions to these optimization problems cannot differ by more than a linear factor in $n$ and a diminishing factor in $\epsilon$.

Let $\Pi(i)$ denote the partition function, that is, $\Pi(i)$ maps from a sample $u_i$ to the partition $\Omega_k$ that it belongs to. The second optimization problem is

$$\begin{aligned} \max_{t_1,\ldots,t_\ell} \quad & \sum_{i=1}^n \left[ \left(1 - \frac{t_{\Pi(i)}}{c(n)}\right) u_i(t_{\Pi(i)}) + \frac{1}{c(n)} \int_0^{t_{\Pi(i)}} u_i(z)\,\mathrm{d}z \right] \\ \text{s. t.} \quad & \sum_{i=1}^n t_{\Pi(i)} \leq c(n), \\ & t_k \geq 0, \quad k = 1, \ldots, \ell. \end{aligned} \tag{14}$$

We claim that for large enough $n$, the solution $t_1^*, \ldots, t_\ell^*$ of problem (14) must satisfy $\sum_{i=1}^n u_i(y_i) \geq \sum_{i=1}^n u_i(t_{\Pi(i)}^*) - \phi\epsilon n$, where $\phi$ is a positive constant. Indeed, for large enough $n$, the second term of the objective function is negligible and the term in parentheses is approximately 1. Therefore, when solving GAME, if we restrict the solution to having the same value if $\Pi(i) = \Pi(j)$, each user's utility diminishes by at most $\epsilon$. Next, we replace the utility function $u_i$ with $\hat{u}_i = \overline{\omega}_{\Pi(i)}$ according to our partition. This leads to:

$$\begin{aligned} \max_{t_1,\ldots,t_\ell} \quad & \sum_{i=1}^n \Bigg[ \left(1 - \frac{t_{\Pi(i)}}{c(n)}\right) \overline{\omega}_{\Pi(i)}(t_{\Pi(i)}) \\ & \qquad + \frac{1}{c(n)} \int_0^{t_{\Pi(i)}} \overline{\omega}_{\Pi(i)}(z)\,\mathrm{d}z \Bigg] \\ \text{subject to} \quad & \sum_{i=1}^n t_{\Pi(i)} \leq c(n), \\ & t_k \geq 0, \quad k = 1, \ldots, \ell. \end{aligned} \tag{15}$$

As a result, for large enough $n$, the problem (15) is an $\epsilon$ perturbation of problem (14) by construction. Both optimization problems are perturbations of the following *deterministic*

---

[2] Recall that for a function class $\Omega : \mathbb{R} \to \mathbb{R}$, a sup-norm $\epsilon$-net is a partition of $\Omega$ into $\Omega_1, \ldots, \Omega_\ell$ such that for every $1 \leq k \leq \ell$ and every pair $u, v \in \Omega_k$ we have that $\sup_{z\in\mathbb{R}} |u(z) - v(z)| < \epsilon$.

problem:

$$\begin{aligned} \max_{t_1,\ldots,t_\ell} \quad & \sum_{k=1}^{\ell} p_k \left[\left(1-\frac{t_k}{c(n)}\right)\overline{\omega}_k(t_k) + \frac{1}{c(n)}\int_0^{t_k} \overline{\omega}_k(z)\,\mathrm{d}z\right] \\ \text{s. t.} \quad & \sum_{k=1}^{\ell} p_k t_k \le c(n)/n, \qquad (16) \\ & t_k \ge 0, \quad k=1,\ldots,\ell, \end{aligned}$$

where $p_k$ is the probability of sampling a utility function in $\Omega_k$ when sampled using $\Pr$.

By standard results for perturbed concave optimization problems [Ber03], the solution to problem (15) are close to the solutions of (14) because they are both perturbations of problem (16). Formally, both of them are random perturbations and we can use perturbation results only for large $n$. This is justified since we eventually take $n$ to infinity in the proof. Therefore, the difference in the aggregate utility is bounded by $n\Delta_1(\epsilon)$ for some $\Delta_1(\epsilon)$ satisfying $\lim_{\epsilon\downarrow 0}\Delta_1(\epsilon)=0$.

The final optimization problem is GAME for the modified utility functions:

$$\begin{aligned} \max_{\hat{y}_1,\ldots,\hat{y}_n} \quad & \sum_{i=1}^{n} \left[\left(1-\frac{\hat{y}_i}{c(n)}\right)\overline{\omega}_{\Pi(i)}(\hat{y}_i) \right. \\ & \qquad \left. + \frac{1}{c(n)}\int_0^{\hat{y}_i} \overline{\omega}_{\Pi(i)}(z)\,\mathrm{d}z\right] \qquad (17) \\ \text{subject to} \quad & \sum_{i=1}^{n} \hat{y}_i \le c(n), \\ & \hat{y}_i \ge 0, \quad i=1,\ldots,n. \end{aligned}$$

By the concavity of each $\overline{\omega}_k$ and following a similar argument, it follows that the aggregate utility cannot decrease by more than a linear factor in $\epsilon$ and $n$ when comparing problems (15) and (17). To summarize, there exists a smooth function $\Delta(\epsilon)$ such that $\lim_{\epsilon\downarrow 0}\Delta(\epsilon)=0$ and:

$$\sum_{i=1}^{n} u_i(y_i) \ge \sum_{i=1}^{n} \hat{u}_i(\hat{y}_i) - \Delta(\epsilon)n. \qquad (18)$$

By Equations (13) and (18), it follows that for every $\epsilon$ and large enough $n$,

$$\text{POA}^{(n)} = \frac{\sum_{i=1}^{n} u_i(y_i^{(n)})}{\sum_{i=1}^{n} u_i(x_i^{(n)})} \ge \frac{\sum_{i=1}^{n} \hat{u}_i(\hat{y}_i^{(n)}) - \Delta(\epsilon)n}{\sum_{i=1}^{n} \hat{u}_i(\hat{x}_i^{(n)}) + \epsilon n}.$$

Since

$$\begin{aligned} & \sum_{i=1}^{n} \hat{u}_i(\hat{x}_i^{(n)}) \ge \sum_{i=1}^{n} \hat{u}_i(c(n)/n) \\ & \qquad\qquad \ge n \min_{i=1,\ldots,n} \hat{u}_i(c(n)/n) > 0 \\ \text{and} \quad & \sum_{i=1}^{n} \hat{u}_i(\hat{x}_i^{(n)}) \ge \sum_{i=1}^{n} \hat{u}_i(\hat{y}_i^{(n)}), \end{aligned}$$

we obtain that for every $\epsilon$ and large enough $n$,

$$\text{POA}^{(n)} \ge \frac{\sum_{i=1}^{n} \hat{u}_i(\hat{y}_i^{(n)})}{\sum_{i=1}^{n} \hat{u}_i(\hat{x}_i^{(n)})} - \overline{\Delta}(\epsilon), \qquad (19)$$

where $\overline{\Delta}(\epsilon)$ is a smooth positive function with $\overline{\Delta}(0)=0$. As we have established, the price of anarchy for finite types of utilities and linearly increasing capacity—*i.e.*, the first term on the right-hand side of Inequality (19)—tends to 1 for every realization of $u_1,\ldots,u_n$. Since $\epsilon$ can be chosen arbitrarily small, we conclude that the loss of efficiency tends to 0 almost surely. ■

To validate the results, we simulate the loss of efficiency for different numbers of users. For scalar-modulated linear utility function, Figure 1 shows that the mean loss of efficiency decreases exponentially in the number of users. Likewise, Figure 2 shows the simulated loss of efficiency for a type of modulated non-linear utility functions. Since there is no closed-form solution for these utility functions, we resort to generic optimization algorithms, which cause prohibitively long simulation times for large numbers of variables. For this reason, we restrict the number of users to be small.

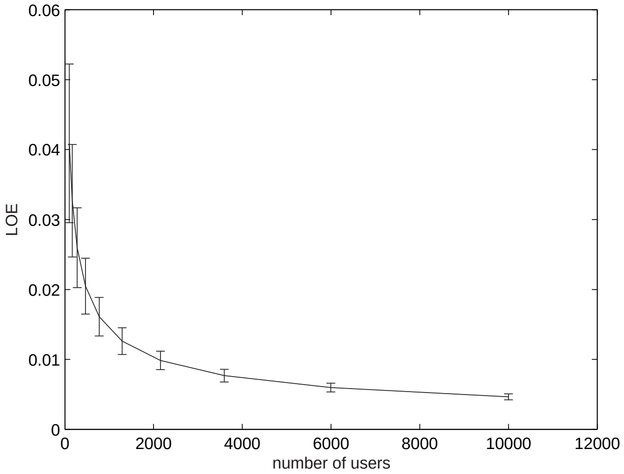

Fig. 1. LOE versus $n$ for $S_i\,x$ utility functions and $S_i$ sampled uniformly in $(0,1)$. The error bars have width equal to one standard deviation.

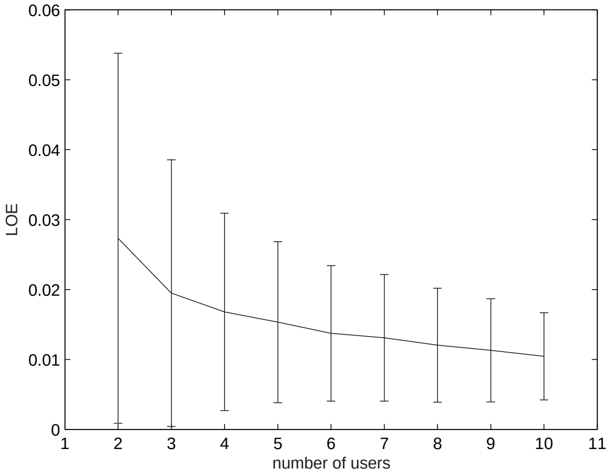

Fig. 2. LOE versus $n$ for $S_i\,x/(1+x)$ utility functions and $S_i$ sampled uniformly in $(0,1)$.

## IV. SITUATIONS WHERE THE LOSS OF EFFICIENCY DOES NOT TEND TO ZERO

In this section, we show that the loss of efficiency is bounded away from 0 with a positive probability for certain classes of distributions for the random utility functions. In particular, we consider the case of linear utility functions $U_i(x) = S_i\, x$, with slopes $S_i$ drawn i.i.d. from some class of distributions with infinite support. For this task, we first relate the loss of efficiency to a ratio of order statistics, then apply results about the asymptotic distribution of order statistics. The theory of order statistics is treated in [DN03] and [LLR83] among other places.

*Definition 4.1:* Let $\mathcal{S}^{(n)} \triangleq \{S_1, \ldots, S_n\}$. The order statistics $\alpha_1(\mathcal{S}^{(n)})$ and $\alpha_2(\mathcal{S}^{(n)})$ are the largest and second-largest elements of $\mathcal{S}^{(n)}$. For brevity sake, we will shorten $\alpha_i(\mathcal{S}^{(n)})$ to $\alpha_i^{(n)}$.

It can be verified that when every $S_i$ has unbounded support, the sequences of highest slopes, $\alpha_1(\mathcal{S}^{(n)})$, $\alpha_2(\mathcal{S}^{(n)})$, and the sequence of shadow prices $\lambda(\mathcal{S}^{(n)})$ do not converge as $n$ tends to infinity. This prompts us to study and describe the asymptotic behavior of the efficiency loss in terms of the ratio of the two highest slopes: $\alpha_1^{(n)}/\alpha_2^{(n)}$.

*Lemma 4.1 (Bound on LOE as a function of $\alpha_2^{(n)}/\alpha_1^{(n)}$):* Consider a game among $n$ users with fixed utility functions of the form $u_i(x) = s_i\, x,\ s_i > 0$. For $n \geq 2$, the efficiency loss is bounded from below as follows:

$$\text{LOE}^{(n)} \geq \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}} \frac{\left(1 - \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)}{\left(1 + \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)}.$$

*Proof:* The result is algebraic in nature, and will be proved for any $n$ and any realization of the random quantities involved. We let $s_1, \ldots, s_n$ denote realizations of $S_1, \ldots, S_n$. At the outset, observe that for linear utility functions, the aggregate utility at the Nash equilibrium is highest when there are fewest active users. For any allocation $(\tilde{y}_i)$ that assigns positive allocation only to two users (say 1 and 2) with the highest slopes $\alpha_1^{(n)}$ and $\alpha_2^{(n)}$, we have

$$\alpha_1^{(n)} \tilde{y}_1^{(n)} + \alpha_2^{(n)} \tilde{y}_2^{(n)} \geq \sum_{i=1}^{n} s_i y_i,$$

where $(y_i)$ is the equilibrium allocation for the game with all $n$ users. Hence, the equilibrium aggregate utility among a group of users with slopes $s_1, \ldots, s_n$ is less than the equilibrium aggregate utility among only two users with maximal slopes $\alpha_1^{(n)}$ and $\alpha_2^{(n)}$. In this case, the optimality conditions (Section II-D) yield the set of equations:

$$\begin{aligned}
\left(1 - \tilde{y}_1^{(n)}\right) \alpha_1^{(n)} &= \mu^{(n)}, \\
\left(1 - \tilde{y}_2^{(n)}\right) \alpha_2^{(n)} &= \mu^{(n)}, \\
\tilde{y}_1^{(n)} + \tilde{y}_2^{(n)} &= c.
\end{aligned}$$

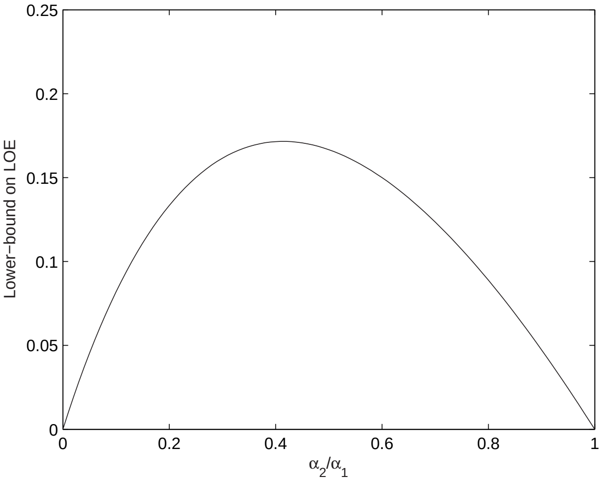

Fig. 3. Lower-bound on LOE as a function of $\alpha_2^{(n)}/\alpha_1^{(n)}$.

From simple calculations, we obtain

$$\begin{aligned}
\tilde{y}_1^{(n)} &= \left(1 + \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)^{-1} c, \\
\tilde{y}_2^{(n)} &= \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}} \left(1 + \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)^{-1} c.
\end{aligned}$$

From the observation at the outset and the definition of efficiency loss, it follows that

$$\begin{aligned}
\text{POA}^{(n)} &\leq \frac{1 + \left(\frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)^2}{1 + \left(\frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)}, \quad \text{for all } n \geq 2, \\
\iff \text{LOE}^{(n)} &\geq \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}} \frac{\left(1 - \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)}{\left(1 + \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)}, \quad \text{for all } n \geq 2. \qquad (20)
\end{aligned}$$

■

The lower-bound on the loss of efficiency from Equation (20) is plotted in Figure 3.

*Corollary 4.2 (Necessary condition for zero LOE):* A necessary condition for the loss of efficiency to tend almost surely to 0 is:

$$\liminf_{n \to \infty} \Pr\left(\frac{\alpha_2^{(n)}}{\alpha_1^{(n)}} = 0 \text{ or } 1\right) = 1.$$

Our next aim is to derive asymptotic characteristics of the ratio $\alpha_2^{(n)}/\alpha_1^{(n)}$ using the theory of order statistics. The largest and second-largest slopes, $\alpha_1^{(n)}$ and $\alpha_2^{(n)}$, are said to be the first and second maximal *order statistics* of the set $\mathcal{S}^{(n)}$. Given the deterministic sequences $\{l_n \mid l_n > 0\}$ and $\{m_n\}$, we define the *linearly normalized order statistics* [LLR83]:

$$\rho_1^{(n)} \triangleq l_n\, (\alpha_1^{(n)} - m_n), \qquad (21a)$$

$$\rho_2^{(n)} \triangleq l_n\, (\alpha_2^{(n)} - m_n). \qquad (21b)$$

These linear transformations are indispensable in order to avoid degenerate distributions as $n \to \infty$. The following theorem, due to Gnedenko, describes the asymptotic distribution of the suitably normalized maximal order statistic.

*Theorem 4.3 (Extremal types, [LLR83]):* Let $S_i$ be independent random variables drawn from a distribution $F(z)$ and $\alpha_1^{(n)} = \max(S_1, \ldots, S_n)$. If there exist deterministic sequences $\{l_n \mid l_n > 0\}$ and $\{m_n\}$ such that the normalized maximal order statistic $\rho_1^{(n)}$ converges in distribution, *i.e.*,

$$\Pr\left(\rho_1^{(n)} \le z\right) \xrightarrow[n\to\infty]{} G(z),$$

for some non-degenerate $G$, then $G$ belongs to one of the following three families of extreme value distributions,[3] for some $\theta > 0$:

$$\begin{aligned} &\text{(Gumbel)} \quad G_1(z) = e^{-e^{-z}}, \quad z \in \mathbb{R}, \\ &\text{(Fréchet)} \quad G_2(z;\theta) = \begin{cases} e^{-z^{-\theta}}, & z > 0, \\ 0, & z \le 0, \end{cases} \\ &\text{(Weibull)} \quad G_3(z;\theta) = \begin{cases} e^{-(-z)^{\theta}}, & z \le 0, \\ 1, & z > 0. \end{cases} \end{aligned}$$

Furthermore, we say that $F(z)$ belongs to the domain of attraction of the corresponding extreme value distribution.

The next theorem gives the necessary and sufficient conditions for a cumulative probability distribution to belong to the domain of attraction of each of the three extreme value distributions.

*Theorem 4.4 (Domains of attraction, [LLR83]):* Let $F(z)$ be the distribution function of the sequence of i.i.d. random variables $\{S_i\}$. Let us denote the upper-bound of the support by $z_F \triangleq \sup\{z \mid F(z) < 1\}$. Let us write $F \in \mathcal{D}(G_i)$ when $F$ belongs to the domain of attraction of $G_i$, $i = 1, 2, 3$. Then,

- $F \in \mathcal{D}(G_1)$ if and only if there exists a strictly positive function $g(t)$ such that
$$\lim_{t \uparrow z_F} \frac{1 - F(t + g(t)z)}{1 - F(t)} = e^{-z}, \text{ for all } z \in \mathbb{R}. \tag{22}$$
- $F \in \mathcal{D}(G_2)$ if and only if $z_F = \infty$ and there exists $\theta > 0$ such that
$$\lim_{t \to \infty} \frac{1 - F(tz)}{1 - F(t)} = z^{-\theta}, \text{ for all } z > 0. \tag{23}$$
- $F \in \mathcal{D}(G_3)$ if and only if $z_F < \infty$ and there exists $\theta > 0$ such that
$$\lim_{t \downarrow 0} \frac{1 - F(z_F - tz)}{1 - F(z_F - t)} = z^{\theta}, \text{ for all } z > 0. \tag{24}$$

Observe that slopes $S_i$ drawn from a distribution $F \in \mathcal{D}(G_3)$ must be bounded from above ($z_F < \infty$); hence, Theorem 3.2 holds. Furthermore, the Gumbel-type distributions have a light tail (Equation (22)), whereas Fréchet-type distributions have a heavy tail (Equation (23)).

*Example 4.1 (Pareto distribution):* Consider the Pareto distribution function $F(z) = 1 - \kappa\, z^{-\theta}$, with parameters $\theta, \kappa > 0$, and with support $[\kappa^{1/\theta}, \infty)$. If $S_i$ has a Pareto distribution, then it is easily verified that the distribution of utility functions $U_i(x) = S_i\, x$ does not satisfy Assumption 2.2 since $\nu = \infty$. If we set $l_n = (\kappa\, n)^{-1/\theta}$ and $m_n = 0$, we obtain [LLR83]:

$$\Pr\left((\kappa\, n)^{-1/\theta}\, \alpha_1^{(n)} \le z\right) \xrightarrow[n\to\infty]{} G_2(z;\theta).$$

In other words, $F \in \mathcal{D}(G_2)$.

For a fixed parameter $\theta > 0$, which is suppressed to lighten notation, we can derive the following limiting distributions characteristic of the Fréchet extreme value distribution [LLR83]:

$$F_{\rho_1}(z_1) = e^{-z_1^{-\theta}}, \quad z_1 > 0,$$

and the following joint, marginal, and conditional probability density functions [LLR83]:

$$\begin{aligned} f_{\rho_1,\rho_2}(z_1, z_2) &= \theta^2 (z_1\, z_2)^{-1-\theta} e^{-z_2^{-\theta}}, \quad 0 < z_2 < z_1, \\ f_{\rho_1}(z_1) &= \theta\, z_1^{-1-\theta}\, e^{-z_1^{-\theta}}, \quad z_1 > 0, \\ f_{\rho_2|\rho_1}(z_2, z_1) &= \theta z_2^{-1-\theta} e^{-z_2^{-\theta}} e^{z_1^{-\theta}}, \quad 0 < z_2 < z_1. \end{aligned} \tag{25}$$

It also follows that

$$F_{\rho_2|\rho_1}(z_2, z_1) = e^{-z_2^{-\theta}} e^{z_1^{-\theta}}. \tag{26}$$

By conditioning and using the above relations, we can bound from below the probability that $\alpha_2^{(n)}/\alpha_1^{(n)}$ is between $\epsilon$ and $1 - \epsilon$ in the limit $n \to \infty$, for $\epsilon \in (0, 1/2)$. Since $\rho_i^{(n)} = (\kappa\, n)^{-1/\theta}\, \alpha_i^{(n)}$, for $i = 1, 2$, we have

$$\begin{aligned} &\liminf_{n\to\infty} \Pr\left(\epsilon < \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}} < 1 - \epsilon\right) \\ &= \liminf_{n\to\infty} \Pr\left(\epsilon < \frac{\rho_2^{(n)}}{\rho_1^{(n)}} < 1 - \epsilon\right) \\ &= \liminf_{n\to\infty} \int_0^\infty \Pr\left(\epsilon z_1 < \rho_2^{(n)} \le (1-\epsilon) z_1 \mid \rho_1^{(n)} = z_1\right) \\ &\qquad \mathrm{d}\big(\Pr(\rho_1^{(n)} = z_1)\big) \\ &\text{(by Fatou's Lemma)} \\ &\ge \int_0^\infty \left[F_{\rho_2|\rho_1}((1-\epsilon)z_1, z_1) - F_{\rho_2|\rho_1}(\epsilon z_1, z_1)\right] f_{\rho_1}(z_1)\, \mathrm{d}z_1 \\ &\text{(by Equations (26) and (25))} \\ &\ge \int_0^\infty \left[e^{-(1-\epsilon)^{-\theta} z_1^{-\theta}} - e^{-\epsilon^{-\theta} z_1^{-\theta}}\right] e^{z_1^{-\theta}} \theta z_1^{-1-\theta} e^{-z_1^{-\theta}}\, \mathrm{d}z_1. \end{aligned}$$

Since the above integrand is positive for $0 < \epsilon < 1/2$, we conclude that in the limit as $n \to \infty$, the ratio $\alpha_2^{(n)}/\alpha_1^{(n)}$ is bounded away from both 0 and 1 with some positive probability. In this case, by Corollary 4.2, the expected efficiency loss is bounded away from 0.

To obtain numerical estimates for the asymptotic lower-bound on the loss of efficiency, we rely on simulations results such as Figure 4. We note that the standard deviation of the loss of efficiency does not decrease to 0. This implies that even if the number of users is large, significant variation in the loss of efficiency is expected.

*Example 4.2 (Cauchy distribution):* The preceding results for Pareto distribution readily extend to the case of one-sided Cauchy distribution, where $F(z) = \frac{2}{\pi}\arctan(z)\mathbf{1}_{[z>0]}$, with $l_n = \tan\left(\frac{\pi}{n}\right)$ and $m_n = 0$ [LLR83]. In this case, $F \in \mathcal{D}(G_2)$

---

[3] Also known as max-stable distributions.

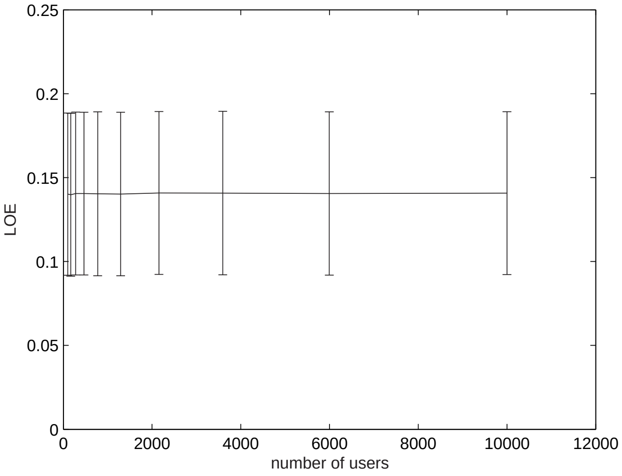

Fig. 4. LOE versus $n$ for $S_i\,x$ utility functions and $S_i$ sampled with Pareto distribution ($\theta = 2$).

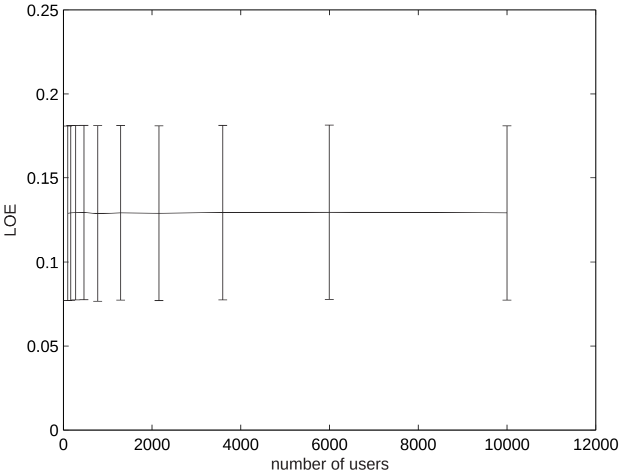

Fig. 5. LOE versus $n$ for $S_i\,x$ utility functions and $S_i$ sampled with Cauchy distribution ($\theta = 1$).

with the parameter $\theta = 1$. Similarly to the Pareto distribution, the loss of efficiency does not converge to 0, nor its empirical standard deviation (see Figure 5).

In fact, for every Fréchet-type random variable (with heavy-tail distribution), we can find a linear transformation with $m_n = 0$ [DN03]. Therefore, the same result as for Pareto distribution example holds, which we summarize in the following corollary.

*Corollary 4.5 (Non-zero asymptotic loss of efficiency):* For scalar-modulated linear utility functions of the form $U_i(x) = S_i\,x$, with $S_i$ drawn i.i.d. from some distribution $F \in \mathcal{D}(G_2)$, the loss of efficiency does not tend to 0 almost surely as $n$ increases to infinity.

The proof follows the same lines as Example 4.1.

For Gumbel-type random variables (with light-tail distributions), such as exponential and Gaussian random variables, it is not clear if a similar result can be proved. Difficulties arise because the normalizing sequences $m_n$ (*cf.* Equations (21)) increase with $n$ [DN03]. Nonetheless, simulations suggest that even for Gumbel-type random variables, a non-zero loss of efficiency can be expected (Figure 6).

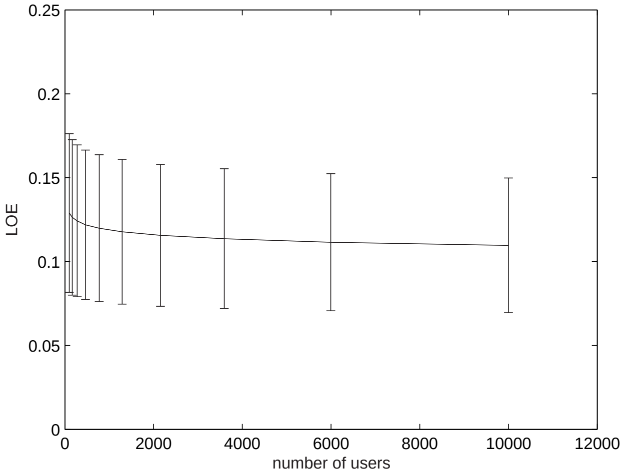

Fig. 6. LOE versus $n$ for $S_i\,x$ utility functions, $S_i$ sampled with exponential distribution.

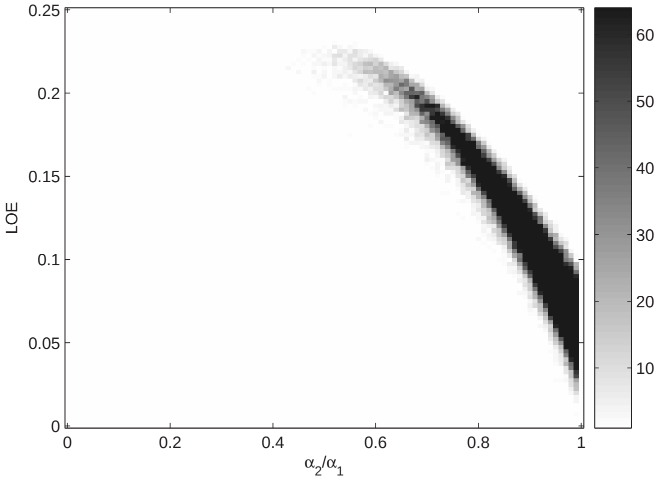

Fig. 7. Empirical joint relative frequency of LOE and $\alpha_2/\alpha_1$ out of $5 \times 10^4$ experiments for $S_i\,x$ utility functions, $S_i$ sampled with exponential distribution, and $n = 10^3$ users.

Figure 7 shows the relative frequency of pairs of values for $(\text{LOE}, \alpha_1/\alpha_2)$ obtained from a simulation where the user slopes $S_i$ are drawn at random according to the exponential distribution $F(z) = (1 - e^{-z})\,\mathbf{1}_{[z>0]}$. The distinctions between Gumbel-type (*e.g.*, exponential, Gaussian) and Fréchet-type (*e.g.*, Pareto, Cauchy) random variables is further emphasized by inspecting Figures 7, 8, and 9. They also highlight the empirical difference in the distribution of $\alpha_2/\alpha_1$ for various distributions of utility functions. For Fréchet-type distributions, there is a noticeably broader range of values taken by $\alpha_2/\alpha_1$. In Figures 7, 8 and 9, we can also clearly observe the outline of the lower-bound plotted in Figure 3.

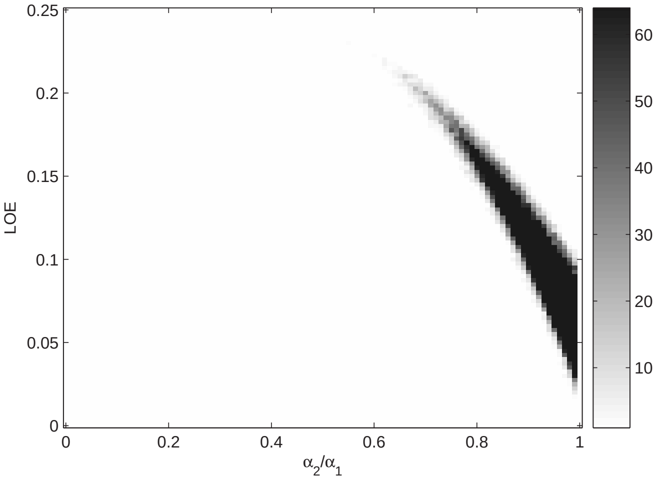

Fig. 8. Empirical joint relative frequency of LOE and $\alpha_2/\alpha_1$ out of $5 \times 10^4$ experiments for $S_i\, x$ utility functions, $S_i$ sampled with Gaussian distribution, and $n = 10^3$ users.

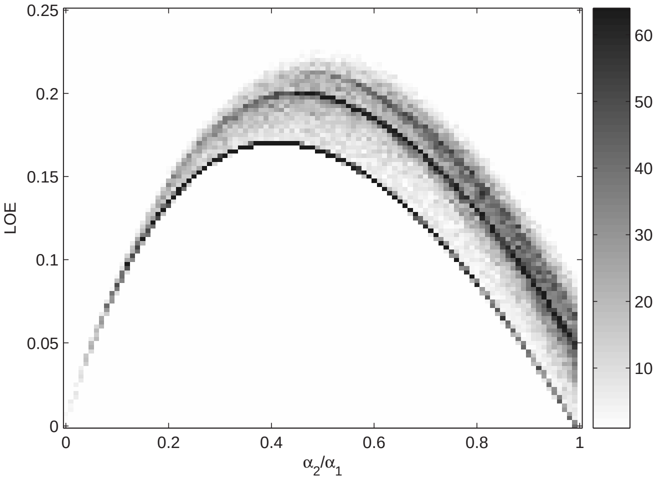

Fig. 9. Empirical joint relative frequency of LOE and $\alpha_2/\alpha_1$ out of $5 \times 10^4$ experiments for $S_i\, x$ utility functions, $S_i$ sampled with Pareto distribution, and $n = 10^3$ users.

## V. NETWORKS

In this section, we give an analogue of Corollary 3.3 extended to networks with arbitrary topology. We constrain the users' utility functions to be linear for convenience of analysis. The case of linear utility functions is noteworthy because it lies at the "boundary" of the class of concave functions of Assumption 2.1. The set of linear utility functions is also rich enough to present situations where the efficiency loss is maximal (*cf.* [JT04]) and asymptotically non-negligible (Section IV).

### A. Network model

We model a network as a simple undirected graph[4] $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. The set of nodes $\mathcal{N}$ comprises the possible sources and destinations. The links of the network compose the set of edges $\mathcal{E}$. To every link $e_k \in \mathcal{E}$ is associated its capacity $c_k$[5].

Consider a set of $n$ deterministic users of the network: $\{1, \ldots, n\}$. Let $\vec{z} = (z_{i,k}) \in \mathbb{R}_+^{|\mathcal{E}|\cdot n}$ denote an allocation where user $i$ receives on link $e_k$ a non-negative amount $z_{i,k}$ of commodity. We are exclusively interested in feasible allocations, *i.e.*, allocations satisfying the hard capacity constraints: $\sum_{i=1}^{n} z_{i,k} \le c_k$ for all $e_k \in \mathcal{E}$. For this section, we assume that each user belongs to one of a *finite* number of *types* of users, denoted by $\tau_1, \ldots, \tau_m$. In other words, the sets $\tau_1, \ldots, \tau_m$ partition the set of users $\{1, \ldots, n\}$. We write $i \in \tau_j$ when user $i$ belongs to type $\tau_j$ and we let $|\tau_j|$ denote the number of users of type $\tau_j$. Moreover, we assume that all users belonging to the same type are *identical* expect for their labels. Each type of users $\tau_j$ is (completely) characterized by a function $w_j : \mathbb{R}_+^{|\mathcal{E}|\cdot n} \to \mathbb{R}_+$, which is the utility function for the members of $\tau_j$. Under an allocation $\vec{z}$, user $i$ of type $\tau_j$ has the following scalar-modulated *linear* utility function:

$$w_j(i; \vec{z}) = s(\tau_j) \cdot \left( \min_{e_k \in \pi_j} z_{i,k} \right),$$

where $\pi_j$ is a path in $\mathcal{G}$ and $s(\tau_j) > 0$. Observe that the utility function captures the notion that the amount of flow that user $i$ can dispatch is limited to the minimum allocated commodity along the path it employs. Observe also that the utility function also captures the notion that every user of type $\tau_j$ employs the same *fixed* path $\pi_j$ in the network[6]. In the network context, we say that user $i$ of type $\tau_j$ is *active* if its effective flow is positive, *i.e.*,

$$\min_{e_k \in \pi_j} z_{i,k} > 0.$$

By extension, a type of users $\tau_j$ is active if there is an active user $i \in \tau_j$.

An allocation $\vec{x} = (x_{i,k})$ is socially optimal if

$$\sum_{j=1}^{m} \sum_{i \in \tau_j} w_j(i; \vec{x}) \ge \sum_{j=1}^{m} \sum_{i \in \tau_j} w_j(i; \vec{z}),$$

or equivalently:

$$\sum_{j=1}^{m} \sum_{i \in \tau_j} s(\tau_j) \cdot \left( \min_{e_k \in \pi_j} x_{i,k} \right) \ge \sum_{j=1}^{m} \sum_{i \in \tau_j} s(\tau_j) \cdot \left( \min_{e_k \in \pi_j} z_{i,k} \right),$$

for every feasible allocation $\vec{z}$. Let $\tau^*$ denote the set of active types of users under the optimal allocation $\vec{x}$, *i.e.*, $\tau^* = \{\tau_j \mid \tau_j \text{ is active}\}$. Hence, $|\tau^*|$ denotes the number of active types of users.

### B. A market-based allocation mechanism for networks

The market mechanism under study is the same as [JT04], which extends the single-link version of Section III. The mechanism allocates each link's resources proportionally to the bids on that link, except when the sum of the bids is zero. More precisely, every user $i$ submits a bid and request pair

---

[4] Parallel edges and self-loops are not allowed.

[5] For the sake of notation, we use the subscript index $k$ to denote the link $e_k$.

[6] We do not consider the question of optimal routing. We also assume that traffic flows are unsplittable.

$(b_{i,k}, r_{i,k}) \in \mathbb{R}^2_+$ to every link $e_k$ in the network. Bids of zero are permitted, so that user $i$ may submit bids of 0 to links outside the path that it needs. Each link $e_k$ then allocates its commodity as follows: for $i = 1, \dots, n$,

$$\text{if } \sum_{\ell=1}^{n} b_{\ell,k} > 0, \text{ then } z_{i,k} = \frac{b_{i,k}}{b_{i,k} + \sum_{\ell \neq i} b_{\ell,k}} c_k,$$

$$\text{otherwise, } z_{i,k} = \begin{cases} r_{i,k}, & \text{if } \sum_{\ell=1}^{n} r_{\ell,k} \leq c_k, \\ 0, & \text{otherwise.} \end{cases}$$

Given the above mechanism and assuming linear utility functions, a price-anticipating user $i$ of type $\tau_j$ and with strategy $(b_i, r_i)$[7] has the following payoff:

$$\text{payoff}_i(b_i, r_i, \vec{b}_{-i}, \vec{r}_{-i}) \triangleq s(\tau_j) \cdot \left( \min_{e_k \in \pi_j} z_{i,k} \right) - \sum_{e_k \in \mathcal{E}} b_{i,k}.$$

In this section, we use the following concept instead of that of Nash equilibria.

*Definition 5.1:* For a fixed $\epsilon > 0$, a strategy profile $(b_1, r_1, \dots, b_n, r_n)$ is an $\epsilon$-equilibrium if

$$\text{payoff}_i(b_i, r_i, \vec{b}_{-i}, \vec{r}_{-i}) \geq \text{payoff}_i(b'_i, r'_i, \vec{b}_{-i}, \vec{r}_{-i}) - \epsilon,$$

for all $(b'_i, r'_i) \in \mathbb{R}^2_+$ and for every user $i = 1, \dots, n$.
We write $\vec{x} = (x_{i,k})$ and $\vec{y} = (y_{i,k})$ to denote, respectively, a socially optimal allocation and an allocation induced by an $\epsilon$-equilibrium strategy profile.

### C. Probabilistic model of random users

Finally, consider a probability space $(\Omega, \mathcal{F}, \Pr)$ over the finite set of functions $\Omega = \{w_1, \dots, w_m\}$, where $w_j$ is the utility function of type $j$ users. We model the users of the network as a sequence of random utility functions $U_1, \dots, U_n$ drawn i.i.d. over this probability space. In the same spirit as Assumption 2.2, we assume that each $U_i$ is distributed such that $\Pr(U_i = w_j) > 0$ for every $w_j$. The loss of efficiency, as in the single-link case, is defined in terms of aggregate utility of all users in the network. However, in the network context, $\epsilon$-equilibria may not be unique. Hence, the efficiency loss is a function of the $\epsilon$-equilibrium allocation $\vec{Y}^{(n)}$. In this paper, we limit our comparison to the *best* $\epsilon$-equilibrium outcome and the socially best outcome,

$$\inf_{\epsilon\text{-equilibria}} \mathrm{LOE}(\vec{Y}^{(n)}) \triangleq \inf_{\epsilon\text{-equilibria}} 1 - \frac{\sum_{i=1}^{n} U_i(\vec{Y}^{(n)})}{\sum_{i=1}^{n} U_i(\vec{X}^{(n)})},$$

where the infimum is over all $\epsilon$-equilibrium strategy profiles. In this setting, we show that the loss of efficiency tends to zero almost surely as the number of users tends to infinity. We begin with a lemma.

*Lemma 5.1:* Consider a network with a single-link of capacity $c$ and $n$ deterministic users. For every $\epsilon > 0$, the payoff to a user $i$ of type $\tau_j$ is such that

$$\left| \text{payoff}_i(b_i, \vec{b}_{-i}) - \text{payoff}_i(\tilde{b}_i, \vec{b}_{-i}) \right| \leq \epsilon,$$

for every $\tilde{b}_i \in \mathbb{R}_+$, provided that

1) the strategy profile $(b_i, \vec{b}_{-i})$ is such that $\sum_{\ell \neq i} b_\ell > 0$, and that price $\mu$ of the link is

$$\sum_{\ell=1}^{n} b_\ell / c = \left( 1 - \frac{1}{|\tau_j|} \right) s(\tau_j), \tag{27}$$

2) and the number of users of type $\tau_j$ is such that

$$|\tau_j| \geq \max \left( s(\tau_j),\ c/\epsilon + 1,\ (s(\tau_j) c/\epsilon)^{1/3} + 1 \right).$$

*Proof:* The result is algebraic. Note that $\sum_{\ell \neq i} b_\ell > 0$ implies that the requests $r_1, \dots, r_n$ have no effect on the outcome. The payoff to user $i$ of type $\tau_j$ is

$$\text{payoff}_i = s(\tau_j) \frac{b_i}{b_i + \sum_{\ell \neq i} b_\ell} c - b_i.$$

If user $i$ bids $b_i + \delta$, for some $\delta > 0$, its payoff becomes

$$\begin{aligned} \text{payoff}_i^{+\delta} &= s(\tau_j) \frac{b_i + \delta}{b_i + \delta + \sum_{\ell \neq i} b_\ell} c - (b_i + \delta) \\ &= s(\tau_j) \frac{b_i + \delta}{\delta/c + \mu} - (b_i + \delta) \\ &\leq s(\tau_j) \frac{b_i}{\mu} - b_i + s(\tau_j) \frac{\delta}{\delta/c + \mu} - \delta. \end{aligned}$$

Suppose, on the one hand, that $\delta > c$, then

$$\begin{aligned} |\tau_j| \geq s(\tau_j) \quad &\Rightarrow \quad \frac{s(\tau_j)}{|\tau_j|} \leq 1 < \frac{\delta}{c} \\ &\Rightarrow \quad s(\tau_j) \leq \frac{\delta}{c} + s(\tau_j) - \frac{s(\tau_j)}{|\tau_j|} = \frac{\delta}{c} + \mu \\ &\Rightarrow \quad s(\tau_j) \frac{\delta}{\delta/c + \mu} - \delta < 0, \end{aligned}$$

which implies that $\text{payoff}_i^{+\delta} < \text{payoff}_i$. Suppose, on the other hand, that $\delta \in (0, c]$, then, for every $\epsilon > 0$,

$$\begin{aligned} |\tau_j| \geq c/\epsilon + 1 \quad &\Rightarrow \quad \epsilon \geq \frac{c}{|\tau_j| - 1} \geq \frac{\delta}{|\tau_j| - 1} \geq \left( \frac{|\tau_j|}{|\tau_j| - 1} - 1 \right) \delta \\ &\Rightarrow \quad \epsilon \geq \left( \frac{s(\tau_j)}{\mu} - 1 \right) \delta \geq s(\tau_j) \frac{\delta}{\delta/c + \mu} - \delta, \end{aligned}$$

which implies that $\text{payoff}_i^{+\delta} \leq \text{payoff}_i + \epsilon$.

Next, if user $i$ bids $b_i - \delta$, for some $\delta \in (0, b_i]$, its payoff becomes

$$\begin{aligned} \text{payoff}_i^{-\delta} &= s(\tau_j) \frac{b_i - \delta}{b_i - \delta + \sum_{\ell \neq i} b_\ell} c - (b_i - \delta) \\ &= s(\tau_j) \frac{b_i - \delta}{\mu - \delta/c} - (b_i - \delta) \\ &\leq s(\tau_j) \frac{b_i}{\mu} - b_i + s(\tau_j) \frac{b_i}{\mu} \frac{\delta/c}{\mu - \delta/c} - s(\tau_j) \frac{\delta}{\mu - \delta/c} + \delta. \end{aligned}$$

---

[7] We use the notation $b_i$ for the vector $(b_{i,k})_{e_k \in \mathcal{E}}$ and $\vec{b}_{-i}$ for the strategy profile all users other than user $i$; likewise for $r_i$.

Therefore,

$$\begin{aligned} \text{payoff}_i^{-\delta} - \text{payoff}_i &\le \frac{\delta}{\mu c-\delta}\left(\frac{s(\tau_j)}{\mu} b_i - s(\tau_j)c + (\mu c - \delta)\right) \\ &\le \frac{\delta}{\mu c - \delta}\left(\frac{|\tau_j|}{|\tau_j|-1} b_i - \frac{s(\tau_j)c}{|\tau_j|} - \delta\right)\\ &\le \frac{1}{|\tau_j|-1}\left(b_i - \frac{s(\tau_j)c}{|\tau_j|} + \frac{b_i}{|\tau_j|-1} - \delta\right)\\ &\le \frac{1}{|\tau_j|-1}\frac{b_i}{|\tau_j|-1}\\ &\le \frac{1}{|\tau_j|-1}\frac{1}{|\tau_j|-1}\frac{s(\tau_j)c}{|\tau_j|} \le \epsilon, \end{aligned}$$

since $|\tau_j| \ge (s(\tau_j)c/\epsilon)^{1/3} + 1$. We have thus shown that user $i$ cannot improve its payoff by more than $\epsilon$. ■

*Theorem 5.2 (Efficiency loss in networks):* Within the network setting described in this section, for every $\epsilon > 0$, we have

$$\Pr\left(\lim_{n\to\infty} \inf_{\epsilon\text{-equilibria}} \mathrm{LOE}(\vec{Y}^{(n)}) = 0\right) = 1.$$

*Remark 5:* Although $\epsilon$-equilibria and Nash equilibria are not unique in the network game, for large enough $n$, we can construct an $\epsilon$-equilibrium starting from a socially optimal allocation. Since the price-taking competitive equilibrium also gives the socially optimal allocation [Kel97], it somewhat justifies our particular choice of $\epsilon$-equilibrium studied. This price-taking equilibrium is arguably the most natural outcome of the game because additional costs may be incurred to find a more profitable strategy.

Note that this theorem does not subsume those from previous sections due to the fact that we assume that there is a finite number of types of users and these users are limited to having linear utility functions.

*Proof:* We proceed in five steps. In the first step, we describe a symmetric socially optimal solution. In the second step, we construct a candidate strategy profile for an $\epsilon$-equilibrium. In the third and fourth steps, we verify the equilibrium conditions for the constructed strategy profile. In the fifth step, we identify the event for which our claims hold and show that it occurs with high probability.

**Step 1**: Find a particular socially optimal allocation (with symmetry and other properties).

Consider a realization $u_1, \ldots, u_n$ of the linear utility functions. First, an optimal social allocation exists and is the solution to the following linear program:

$$\begin{aligned} \max_{\{x_{i,k}\},\{x_i\}} \quad & \sum_{j=1}^{m}\sum_{i\in\tau_j} s(\tau_j)\, x_i, \\ \text{subject to} \quad & x_i \le x_{i,k}, \quad e_k \in \pi_j, \quad i \in \tau_j, \\ & \sum_{i=1}^{n} x_{i,k} \le c_k, \quad e_k \in \mathcal{E}, \\ & x_{i,k} \ge 0, \quad i = 1,\ldots,n, \quad e_k \in \mathcal{E}, \\ & x_i \ge 0, \quad i = 1, \ldots, n. \end{aligned}$$

Next, observe that there exists a solution such that

- if $i \in \tau_j$, then $x_{i,k} = x_i$, for all $e_k \in \pi_j$,
- $x_{i,k} = 0$ for all $e_k \notin \pi_j$,
- if $i_1$ and $i_2$ belong to the same type $\tau_j$, then $x_{i_1} = x_{i_2}$.

In other words, no user is allocated more resource than necessary, users receive zero allocation on links outside their path, and users of the same type receive the same allocation. Letting $x(\tau_j) = \sum_{i\in\tau_j} x_i$, we can find such a solution by solving the following linear program:

$$\begin{aligned} \max_{x(\tau_1),\ldots,x(\tau_m)} \quad & \sum_{j=1}^{m} s(\tau_j)\, x(\tau_j), \\ \text{subject to} \quad & \sum_{\tau_j : e_k \in \pi_j} x(\tau_j) \le c_k, \quad e_k \in \mathcal{E}, \\ & x(\tau_j) \ge 0, \quad j = 1, \ldots, m, \end{aligned} \tag{28}$$

**Step 2**: Construct a candidate strategy profile for which we will verify the $\epsilon$-equilibrium conditions.

Given such an allocation $(x_{i,k})$ that solves the linear program (28), we can construct the following strategy profile:

$$b_{i,k} = \zeta_k\, x_{i,k}, \quad \text{for every user } i \text{ and link } e_k, \tag{29}$$

$$r_{i,k} = x_{i,k}, \quad \text{for every user } i \text{ and link } e_k, \tag{30}$$

where $\{\zeta_k\}$ satisfies the following constraints:

$$\zeta_k \ge 0, \quad \text{for every link } e_k, \tag{31a}$$

$$\zeta_k = 0, \quad \text{for all } e_k \notin \tau^*, \tag{31b}$$

$$\sum_{e_k\in\pi_j} \zeta_k = \left(1 - \frac{1}{|\tau_j|}\right) s(\tau_j), \quad \text{for all } \tau_j \in \tau^*, \tag{31c}$$

$$\sum_{e_k\in\pi_j} \zeta_k \ge s(\tau_j), \quad \text{for all } \tau_j \notin \tau^*. \tag{31d}$$

We shall use facts from linear programming[8] and the continuity of solutions to linear equations to show the existence of a set $\{\zeta_k\}$ satisfying the constraints (31). Consider the dual of the linear program (28):

$$\begin{aligned} \min_{\zeta_1,\ldots,\zeta_{|\mathcal{E}|}} \quad & \sum_{k=1}^{|\mathcal{E}|} c_k\, \zeta_k, \\ \text{subject to} \quad & \sum_{e_k\in\pi_j} \zeta_k \ge s(\tau_j), \quad j = 1, \ldots, m, \\ & \zeta_k \ge 0, \quad \text{for all } e_k \in \mathcal{E}. \end{aligned}$$

Since the primal linear program has an optimal solution, its dual also has an optimal solution. Furthermore, the optimal solution satisfies the primal feasibility, dual feasibility, and complementary slackness conditions. It follows that there exist vectors $\vec{x}$ and $\vec{\zeta}$ that solve the following system of equations.

(Primal feasibility)

$$\sum_{\tau_j : e_k\in\pi_j} x(\tau_j) \le c_k, \quad \text{for all } e_k \in \mathcal{E}, \tag{32a}$$

$$x(\tau_j) \ge 0, \quad j = 1, \ldots, m, \tag{32b}$$

(Dual feasibility)

$$\sum_{e_k\in\pi_j} \zeta_k \ge s(\tau_j), \quad j = 1, \ldots, m, \tag{32c}$$

$$\zeta_k \ge 0, \quad \text{for all } e_k \in \mathcal{E}, \tag{32d}$$

---

[8] Necessary results can be found in [Chv83] and the references therein.

(Complementary slackness)

$$\sum_{e_k\in\pi_j}\zeta_k = s(\tau_j), \quad \tau_j \in \tau^*, \tag{32e}$$

$$\zeta_k = 0, \quad \text{for all } e_k \text{ such that } \sum_{\tau_j : e_k\in\pi_j} x(\tau_j) < c_k. \tag{32f}$$

Note that Equations (32e) and (32f) form only part of the complementary slackness conditions.

By perturbing the term $s(\tau_j)$ in Equation (32e) and removing the constraints for $\tau_j \in \tau^*$ in Equations (32c), we obtain

$$\sum_{e_k\in\pi_j}\zeta_k \geq s(\tau_j), \quad \tau_j \notin \tau^*, \tag{33a}$$

$$\sum_{e_k\in\pi_j}\zeta_k = \left(1 - \frac{1}{|\tau_j|}\right) s(\tau_j), \quad \tau_j \in \tau^*. \tag{33b}$$

Let us replace Equations (32c) and (32e) by Equations (33a) and (33b) in the system of equations (32). By continuity of the solution to Equations (32a), there exists a $\delta > 0$ small enough such that for $|\tau_j| > s(\tau_j)/\delta$, there exists a solution $(\vec{x'}, \vec{\zeta'})$ to the new system of equations. Hence, there exists a set of variables $\{\zeta_k\}$ that satisfy the system of equations (31).

Next, let us verify that that the allocation induced by the strategy profile (29)-(30) is socially optimal. Observe that by construction of the candidate strategy profile (Equation (29)), we have

$$\sum_{\ell=1}^{n} r_{\ell,k} = \sum_{\ell=1}^{n} x_{\ell,k} \leq c_k.$$

Hence, in the event that $\sum_{\ell=1}^{n} b_{\ell,k} = 0$, the allocation of each link $e_k$ is

$$z_{i,k} = r_{i,k}\mathbf{1}_{[\sum_{\ell=1}^{n} r_{\ell,k}\leq c_k]} = x_{i,k}, \quad i = 1,\ldots,n.$$

When $\sum_{\ell=1}^{n} b_{\ell,k} \neq 0$, the allocation of each link $e_k$ is

$$z_{i,k} = \frac{b_{i,k}}{b_{i,k} + \sum_{\ell\neq i} b_{\ell,k}} c_k = \frac{\zeta_k\, x_{i,k}}{\sum_{\ell=1}^{n} \zeta_k\, x_{\ell,k}} c_k = x_{i,k}, \quad i = 1,\ldots,n.$$

In sum, by construction of both $b_{i,k}$ and $r_{i,k}$ (Equations (29) and (30)), the allocation resulting from the strategy profile is $(z_{i,k}) = (x_{i,k})$.

Note that the allocation $(z_{i,k})$ induced by the given strategy profile is the same as the optimal allocation $(x_{i,k})$. Hence, active users are the same in both allocations. For the same realization $u_1, \ldots, u_n$, we now proceed to show that the above strategy profile is a Nash equilibrium by showing that the payoffs to both inactive and active users cannot be improved unilaterally.

**Step 3**: Verify equilibrium conditions for active users.

Consider an active user $i$ of type $\tau_j$. Suppose that $|\tau_j|$ satisfies the assumption of Lemma 5.1. Suppose, on the contrary, that user $i$ plays $(\tilde{b}_{i,k})$ instead of $(b_{i,k})$ and improves its payoff by more than some $\epsilon > 0$, that is,

$$s(\tau_j) \min_{e_k\in\pi_j}\left(\frac{\tilde{b}_{i,k}}{\tilde{\zeta}_k}\right) - \sum_{e_k\in\pi_j} \tilde{b}_{i,k}$$
$$> s(\tau_j) \min_{e_k\in\pi_j}\left(\frac{b_{i,k}}{\zeta_k}\right) - \sum_{e_k\in\pi_j} b_{i,k} + \epsilon,$$

where $\tilde{\zeta}_k$ denotes the resultant price $\left(\sum_{\ell\neq i} b_{\ell,k} + \tilde{b}_{i,k}\right)/c_k$ of link $e_k$. Clearly,

$$\frac{\sum_{e_k\in\pi_j}\tilde{b}_{i,k}}{\sum_{e_k\in\pi_j}\tilde{\zeta}_k} \geq \min_{e_k\in\pi_j}\left(\frac{\tilde{b}_{i,k}}{\tilde{\zeta}_k}\right),$$

moreover, by construction, we have

$$\min_{e_k\in\pi_j}\left(\frac{b_{i,k}}{\zeta_k}\right) = \frac{\sum_{e_k\in\pi_j} b_{i,k}}{\sum_{e_k\in\pi_j}\zeta_k}$$

Hence, we get a contradiction with Lemma 5.1.

In addition, user $i$ cannot improve its payoff by adjusting bids among the links on its path while keeping the sum $\sum_{e_k\in\pi_j} b_{i,k}$ fixed. The reason is that if some bid $b_{i,k}$ decreases, the overall flow decreases as well. It is optimal for an active user $i$ to request $r_{i,k}$ as specified in the given strategy profile. If $\zeta_k > 0$, then all requests are ignored. If $\zeta_k = 0$ and all other users request their socially optimal allocation according to $\vec{x}$, then requesting $r_{i,k} > x_{i,k}$ does not increase user $i$'s utility (otherwise, the allocation $\vec{x}$ would not be optimal). Obviously, neither does requesting $r_{i,k} \leq x_{i,k}$ benefit user $i$.

**Step 4**: Verify equilibrium conditions for inactive users.

Consider an inactive user $i \in \tau_j$, and a strategy profile where user $i$ alone deviates from $(b_{i,k})$ to $(\tilde{b}_{i,k})$. Its new payoff becomes

$$\text{payoff}_i(\tilde{b}_i, \vec{b}_{-i}) = s(\tau_j)\left(\min_{e_k\in\pi_j}\tilde{z}_{i,k}\right) - \sum_{e_k\in\mathcal{E}}\tilde{b}_{i,k},$$

$$\text{where} \quad \tilde{z}_{i,k} = \frac{\tilde{b}_{i,k}}{\tilde{b}_{i,k} + \sum_{\ell\neq i} b_{\ell,k}} c_k.$$

Observe that user $i$'s optimal bidding strategy must be such that $\tilde{z}_{i,k} = \tilde{z}_i$ for a constant $\tilde{z}_i$ for all $e_k \in \pi_j$ and that $\tilde{b}_{i,k} = 0$ for all $e_k \notin \pi_j$. Hence,

$$\text{payoff}_i(\tilde{b}_i, \vec{b}_{-i}) \leq s(\tau_j)\,\tilde{z}_i - \sum_{e_k\in\pi_j}\tilde{z}_i\,\tilde{\zeta}_k,$$

where $\tilde{\zeta}_k$ is the new price of link $e_k$. Since user $i$'s bids $(\tilde{b}_{i,k})$ are non-negative, we have $\tilde{\zeta}_k \geq \zeta_k$. Finally, by Equation (31d), we find that

$$\text{payoff}_i(\tilde{b}_i, \vec{b}_{-i}) \leq \tilde{z}_i\left(s(\tau_j) - \sum_{e_k\in\pi_j}\zeta_k\right) \leq 0.$$

It follows that an inactive user $i$ cannot improve its payoff unilaterally. By the same reasoning as for active users, it is optimal for each inactive user $i$ to request $r_{i,k} = 0$ as in the candidate strategy profile.

**Step 5**: Identify the probabilistic event for which Steps 2, 3, and 4 hold.

Recall that $|\tau_1|, \dots, |\tau_m|$ are random quantities that depend on the random utility functions $U_1, \dots, U_n$. For a fixed $\delta > 0$ and every $n$, we define the event $E_n = \big\{ |\tau_j|(U_1, \dots, U_n) \leq 1/\delta \text{ for some } j \in \{1, \dots, m\}\big\}$. Let $q \triangleq \max_{j=1,\dots,m} \Pr(U_i = w_j)$. By Lemma 3.1, we have

$$\begin{aligned}\sum_{n\geq 0} \Pr(E_n) &\leq \sum_{n\geq 0} m \max_{j=1,\dots,m} \Pr(|\tau_j|(U_1, \dots, U_n) \leq 1/\delta) \\ &\leq m \sum_{n\geq 0} \sum_{k=0}^{\lfloor 1/\delta \rfloor} \binom{n}{k} q^k (1-q)^{n-k} \\ &\leq m \sum_{n=0}^{\lfloor 1/(\delta q) \rfloor} \binom{n}{k} q^k (1-q)^{n-k} \\ &\quad + m \sum_{n \geq \lceil 1/(\delta q) \rceil} \exp\left\{ -2 \, \frac{(nq - \lfloor 1/\delta \rfloor)^2}{n} \right\} < \infty.\end{aligned}$$

Therefore, by the Borel-Cantelli Lemma, the probability that infinitely many events $E_n$ occur is 0. Therefore, for large enough $n$, we have $|\tau_j| > 1/\delta$ with probability 1 for every type $\tau_j$. We have shown that this event guarantees that there is an optimal allocation that maps to an $\epsilon$-equilibrium with the same allocation. The claimed result follows. ■

## VI. RELATED WORKS IN ECONOMICS

The subject of games with many players, and particularly markets with many traders, has been extensively studied in economic theory and game theory. The idea that price-taking behavior is characteristic of large markets has been around for a long time. It is often observed that non-cooperative equilibria of markets tend to be inefficient in the presence of few participants, but efficient when there are many comparable participants. The notion of perfect competition among a large number of producers dates back to Cournot's 1838 work [Cou38]. In this section, we situate our contributions with respect to some related works.

There are predominantly two approaches for modeling a large number of participants. The first approach assumes a continuum of participants. This does not give any information about what happens with few or moderately many users. In a market with a continuum of traders, *e.g.*, represented by an interval of the real line, Dubey *et al.* [DMCS80] show that, under conditions of convexity of strategy sets, continuity of outcomes with respect to strategies, anonymity of traders, and aggregation of other traders' strategies, every non-cooperative equilibrium of continuum markets is Walrasian, and hence, efficient. However, an important question is the extent to which the continuum-of-participants assumption reflects markets with large, but finite, numbers of participants. This question is addressed in [HM72], [DMCS80], [Gre80], [Gre84], and [Car03], among other works. Hildenbrand and Mertens [HM72], Dubey *et al.* [DMCS80], and Green [Gre84] show that the limit of a sequence of equilibria of finite markets is an equilibrium in a continuum market if the equilibrium correspondence is upper hemicontinuous. However, there exist sequences of finite games whose Nash equilibria differ greatly from those of the corresponding continuum-of-agents games [Car03]. We do not rely on the continuum of participants idealization. We consider a countable number of participants, whereas using the continuum model inherently ensures that each individual participant is strategically insignificant [Aum64].

The second approach involves replicating a finite set of participants, without explicit regard for capturing heterogeneity. In a sequence of replica markets, where increasingly large market are created by adjoining a copy of an original finite market, individual rationality is compatible with social rationality (*i.e.*, price-taking behavior) in the limit [RP76], [Gre80], [Rob80]. Green [Gre80] shows that if a sequence of equilibria converges, the limit is a price-taking equilibrium of the corresponding continuum representation. In dynamic markets, however, Green [Gre80] gives counter-examples where non-cooperative equilibria fail to be efficient even with a large number of participants. Roberts [Rob80] examines when individual rationality is compatible with social rationality and price-taking behavior. In more general sequences of economies (not necessarily generated by replication), Roberts and Postlewaite [RP76] show that the incentive for one participant to deviate from price-taking behavior becomes arbitrarily small in the limit, as long as each participant's relative endowment become arbitrarily small as well.

For finite markets, Postlewaite and Schmeidler [PS78] show that under additional conditions on the initial distribution of resources, any allocation resulting from a Nash equilibrium is approximately efficient for a large enough economy. Our work differs from [PS78] in that we relax their restrictions on the initial distribution of resources and that we consider efficiency with respect to *aggregate* utility.

The present work offers a new take on an old problem. The resource allocation problem that we consider is a special case of markets (or exchange economies) described in the literature. In terms of motivation, modeling, and analysis, our work is very different. Our emphasis lies in modeling heterogeneity, and not only the effect of large numbers. We adopt a probabilistic line of analysis. An important distinction is that we consider a random sampling of a countable number of participants with a continuous set of characteristics (*i.e.*, utility functions in our case). This allows us to derive convergence results applicable to finite markets. Our method of sampling random participants is similar in spirit to that of Palfrey and Srivastava [PS86]. They consider an economy with "stochastically replicated" agents possessing random private information, and show that the incentive to conceal one's private information decreases to zero as the number of agents increases. In contrast to our work, part of their results rely on the assumption that each agent's private information is drawn from a *finite* probability space. Furthermore, we also consider the possibility that amount of commodity available varies with the number of users.

## VII. CONCLUSION

In this paper, we studied the efficiency loss in two market mechanisms proposed by Kelly [Kel97] for the allocation of network resources. Not only is the loss of efficiency bounded [JT04], it also converges to zero exponentially under some standard assumptions in the single-link case (Section III). We further showed almost sure convergence of the efficiency loss in the case where the capacity is a function of the number of users. A similar result for the case of elastic capacity is presented in [YM06]. There are, however, cases where the efficiency loss provably does not converge to zero (Section IV). We have established that the loss of efficiency also converges to zero in general network topologies when users have linear utility functions (Section V). Therefore, bigger—more populous—networks are usually more efficient. Analogies can be drawn with works in the economics literature (Section VI).

We conclude that the lack of central regulation, or of cooperation among users, does not result in efficiency loss as long as no single user enjoy the commodity much more than the rest. Interpreted differently, we can also conclude that price-anticipating (*i.e.*, individually optimal) behavior is not considerably more advantageous than price-taking behavior in the presence of many comparable users.

This work is an initial step toward understanding the behavior of large and random competitive networks. Our setup requires combining tools from probability, optimization, and game theory, which is reminiscent of the machine learning methodology. The methods developed here can possibly be adapted to other setups (*e.g.*, selfish routing) to draw similar conclusions. By sampling the utility functions from a *dependent* random process, it is further possible to model *user-response* to a certain network situation. Another natural extension is the inclusion of users that are not self-optimizing. Further important research directions include studying the effect of network topology on the loss of efficiency. It would be interesting to see whether some topologies exhibit a reduced loss by design. Another related problem that may, perhaps, be approached using a probabilistic model are games where routing is part of the strategic decision, as opposed to our assumption of fixed routing given the network topology. Finally, as we observed in experiments, even for linear utility functions modulated by a random variable with heavy-tail distribution, there is significant asymptotic efficiency loss. Characterizing the asymptotic distribution of this efficiency loss is a challenging problem.

## REFERENCES

[AB99] M. Anthony and P. L. Bartlett. *Neural Network Learning: Theoretical Foundations*. Cambridge University Press, 1999.

[Aum64] R. J. Aumann. Markets with a continuum of traders. *Econometrica*, 32:39–50, 1964.

[Ber03] D. P. Bertsekas. *Convex Analysis and Optimization*. Athena Scientific, 2003.

[Car03] G. Carmona. Nash and limit equilibria of games with a continuum of players. Technical Report 0311004, Economics Working Paper Archive at WUSTL, November 2003.

[Chv83] V. Chvátal. *Linear Programming*. W. H. Freeman, 1983.

[Cou38] A. A. Cournot. *Recherches sur les Principes Mathématiques de la Théorie des Richesses*. Hachette, 1838.

[DMCS80] P. Dubey, A. Mas-Colell, and M. Shubik. Efficiency properties of strategic market games: an axiomatic approach. *Journal of Economic Theory*, 1980.

[DN03] H. A. David and H. N. Nagaraja. *Order Statistics*. Wiley-Interscience, 3rd edition, 2003.

[Fri04] E. J. Friedman. A generic analysis of selfish routing. In *Proceedings IEEE Conference on Decision and Control*, 2004.

[Gre80] E. J. Green. Noncooperative price taking in large dynamic markets. *Journal of Economic Theory*, 22:155–182, 1980.

[Gre84] E. J. Green. Continuum and finite-player noncooperative models of competition. *Econometrica*, 52(4):975–93, 1984.

[HG04] B. Hajek and G. Gopalakrishnan. A framework for studying demand in hierarchical networks. Preliminary draft, 2004.

[HM72] W. Hildenbrand and J.F. Mertens. Upper hemi-continuity of the equilibrium set correspondence for pure exchange economies. *Econometrica*, 1972.

[Hoe63] W. Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, 1963.

[JMT05] R. Johari, S. Mannor, and J. Tsitsiklis. Efficiency loss in a network resource allocation game: the case of elastic supply. *IEEE Transactions on Automatic Control*, 50(11):1712–1724, 2005. Also LIDS Technical Report 2605.

[JT04] R. Johari and J. N. Tsitsiklis. Efficiency loss in a network resource allocation game. *Mathematics of Operations Research*, 29(3):407–435, 2004.

[Kel97] F. P. Kelly. Charging and rate control for elastic traffic. *European Transactions on Telecommunications*, 8:33–37, 1997.

[KP99] E. Koutsoupias and C. Papadimitriou. Worst-case equilibria. *Lecture Notes in Computer Science*, 1563:404–413, 1999.

[LLR83] M. R. Leadbetter, G. Lindgren, and H. Rootzén. *Extremes and related properties of random sequences and processes*. Springer-Verlag, 1983.

[PS78] A. Postlewaite and D. Schmeidler. Approximate efficiency of non-Walrasian Nash equilibria. *Econometrica*, 46(1):127–35, 1978.

[PS86] T. R. Palfrey and S. Srivastava. Private information in large economies. *Journal of Economic Theory*, 39:34–58, 1986.

[QYZS03] L. Qiu, Y. R. Yang, Y. Zhang, and S. Shenker. On selfish routing in Internet-like environments. In *Proceedings ACM SIGCOMM*, pages 151–162, 2003.

[Rob80] K. Roberts. The limit points of monopolistic competition. *Journal of Economic Theory*, 22(2):256–278, 1980.

[RP76] D. J. Roberts and A. Postlewaite. The incentives for price-taking behavior in large exchange economies. *Econometrica*, 44(1):115–127, 1976.

[RT02] T. Roughgarden and E. Tardos. How bad is selfish routing? *Journal of the ACM*, 49(2):236–259, 2002.

[Sha76] L. S. Shapley. Noncooperative general exchange. In S. A. Y. Lin, editor, *Theory and Measurement of Economic Externalities*, pages 155–175. Academic Press, 1976.

[She95] S. Shenker. Fundamental design issues for the future internet. *IEEE Journal on Selected Areas in Communication*, 13(7):1176–1188, 1995.

[Shu73] M. Shubik. Commodity money, oligopoly, credit and bankruptcy in a general equilibrium model. *Western Economic Journal*, 1973.

[YM06] J. Y. Yu and S. Mannor. Asymptotics of efficiency loss in competitive market mechanisms. In *Proceedings IEEE INFOCOM*, 2006. To appear.