Efficiency Loss in a Resource Allocation Game: A Single Link in Elastic Supply

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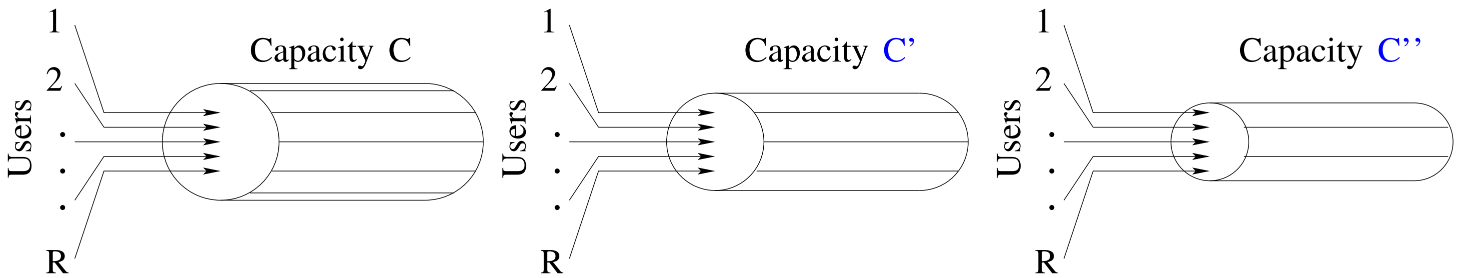
Agenda

Allocation of divisible resources
(e.g., bandwidth in a network)

Simple “mechanisms”
(based on market clearing)

Study efficiency loss, w.r.t. social optimum
(in the presence of selfish users)

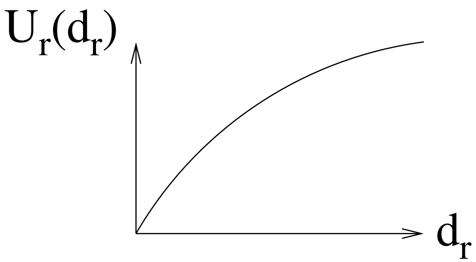
Single link model



Each user:

Rate d_r $\longrightarrow$ utility $U_r(d_r)$

U_r: concave, strictly increasing, nonnegative



Capacity

Users want more BW.

Supplying f units incurs cost \$$C(f)$.

$C(f)$ strictly increasing, strictly convex, nonnegative

U_r and C are measured in dollars.

An efficient allocation maximizes net monetary benefit:

$$\sum_r U_r(d_r) - C\left(\sum_r d_r\right)$$

Our goal: achieve an efficient allocation.

The Social Optimum

SYSTEM:

$$\begin{array}{rl} \text{maximize} & \displaystyle\sum_r U_r(d_r) - C\left(\sum_r d_r\right) \\ \text{subject to} & d_r \geq 0 \quad r = 1, 2, \ldots, R \end{array}$$

Social welfare, net monetary benefit.

Observation: *SYSTEM* has a unique solution.

The control problem

The society wants to get high social welfare.

But U_r are individualistic:

1. Can't know U_r.
2. R can be very large.
3. Users are selfish.

Resource manager - the visible hand.

Need to decide how to divide $C(\sum_r d_r)$ between the users.

A Pricing Mechanism

Define the *price function* $p(f)$ such that $C(f) = \int_0^f p(s)ds$.

$p(f)$ strictly increasing, convex, nonnegative

We use the following mechanism (Kelly, Maulloo, and Tan, 1998):

(1) User r submits a bid w_r.

(2) The mechanism chooses $f(\boldsymbol{w})$ to "clear the market":

$$\sum_r w_r \quad = \quad f(\boldsymbol{w}) \quad \times \quad p(f(\boldsymbol{w}))$$

Revenue = Quantity × Price

User r's rate: $d_r = w_r / p(f(\boldsymbol{w}))$

User r's payoff: $U_r(d_r) - w_r$

Types of games

Bids determine price of BW unit.

Do users anticipate their effect?

1. No (price takers).
2. Yes, but one step lookahead (price anticipators).
3. Yes, longer lookahead.

Users as Price Takers

Suppose users assume price $\mu = p(f(\boldsymbol{w}))$ does not depend on their bid.

Each user optimizes:

$$P_r(w_r, \mu) = U_r\left(\frac{w_r}{\mu}\right) - w_r$$

Theorem 1 (Kelly, Maulloo, and Tan, 1998)

There exist $\boldsymbol{w}$ and μ such that:

1. w_r is optimal for user r given the price μ.
2. $\mu = p(f(\boldsymbol{w}))$.

Further, $\boldsymbol{d} = \boldsymbol{w}/\mu$ is an efficient allocation.

Users as Price Anticipators

Suppose users *know* the price setting procedure.

Let $\mu(\boldsymbol{w})$ be the price for a vector of bids $\boldsymbol{w}$.

If user r changes w_r to w_r' he expects that:
1. $\boldsymbol{w}_{-r} = (w_s, s \neq r)$ will not change.
2. A new price $\mu' = \mu(\boldsymbol{w}_{-r}; w_r')$ will be set.
3. Allocated BW $d_r' = w_r'/\mu'$.
4. The other users will not change their bids.

Optimize:

$$\max_{w_r \geq 0} \; U_r\left(\frac{w_r}{\mu(\boldsymbol{w}_{-r}; w_r)}\right) - w_r$$

Nash Equilibrium

What if all users are price anticipators?

This is now a *game*, where the strategy of user r is the bid w_r.

Interested in Nash equilibria

Theorem 2 *If all users are price anticipators then there exists a Nash equilibrium.*

Proof idea: Show that $w_r/\mu(\boldsymbol{w}_{-r}; w_r)$ is concave in w_r for a fixed $\boldsymbol{w}_{-r}$.
Apply Rosen's existence theorem.

Price Anticipating Users: Efficiency Loss

Theorem 3

The efficiency loss is no more than ≈ 34%:

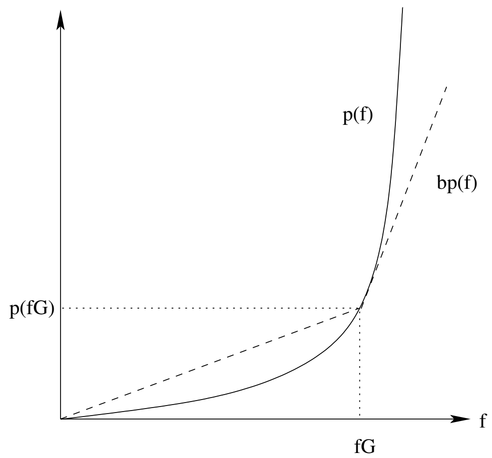
$$\begin{array}{c}\textit{Net monetary benefit}\\ \textit{at Nash equilibrium}\end{array} \geq (4\sqrt{2}-5) \times \begin{array}{c}\textit{Net monetary benefit}\\ \textit{at efficient allocation}\end{array}$$

Furthermore, this bound is tight.

Proof idea:

1. Show $U_r(d_r) = \alpha_r d_r$ on the worst case.
2. $f = 1$.
3. For a fixed p compute worst case ratio.
4. Piecewise linear price functions are enough.
5. Minimize ratio.

The Worst Case



$$
\begin{aligned}
&f = 1 \\
&U_1(d_1) = d_1 \\
&U_r(d_r) \approx (2 - \sqrt{2})d_r \\
&p\text{: } a = 2 - \sqrt{2} \\
&\quad b \to \infty \\
&R \to \infty
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
&d_1^G \to \sqrt{2} - 1 \\
&\sum_{r>1} d_r^G \to 2 - \sqrt{2}
\end{aligned}
$$

Additional Results

- If $C(f) = f^B$, then as $B \to \infty$, efficiency loss $\to$ 25%.
  (Intuition: *as if* the resource is in inelastic supply.)

- Results generalize to network context.

Back to Congestion Control

Faster time scale

Fixed bids w: network protocol allocates rates

Slower time scale

Users observes link prices and adjust bids

Implicit in our model:

(a) Can anticipate the effect of a bid change on prices

(b) Do not anticipate changes (reactions) in bids of other users

Interesting Directions

Dynamics of convergence to equilibrium

Multi-time step look ahead

“Complexity” versus efficiency guarantees

Different information models

Efficiency loss for *generic* cost functions