A Game Theoretic View of Efficiency in Network Resource Allocation

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Agenda

Allocation of divisible resources
(e.g., bandwidth in a network)

Simple “mechanisms”
(based on market clearing)

Study efficiency loss, w.r.t. social optimum
(in the presence of selfish users)

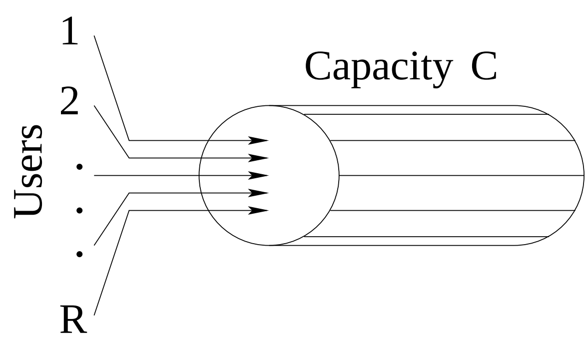
Outline

1. Single link, fixed capacity

2. Network case, fixed capacities

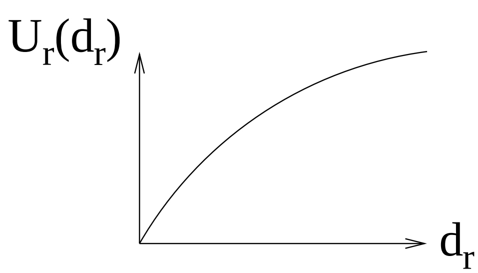
3. Elastic capacities

PART I: Single Link, Fixed Capacity



Rate d_r $\longrightarrow$ utility $U_r(d_r)$

U_r: concave, strictly increasing, nonnegative



The Social Optimum

$$
\begin{aligned}
\text{maximize} \quad & \sum_r U_r(d_r) \\
\text{subject to} \quad & \sum_r d_r \le C \\
& \boldsymbol{d} \ge 0
\end{aligned}
$$

A Pricing Mechanism

User r submits a bid w_r.

Receives bandwidth: $\dfrac{w_r}{w_1 + \cdots + w_R} C$

Example: $w_1 = 2$, $w_2 = 3$; $d_1 = 2C/5$, $d_2 = 3C/5$

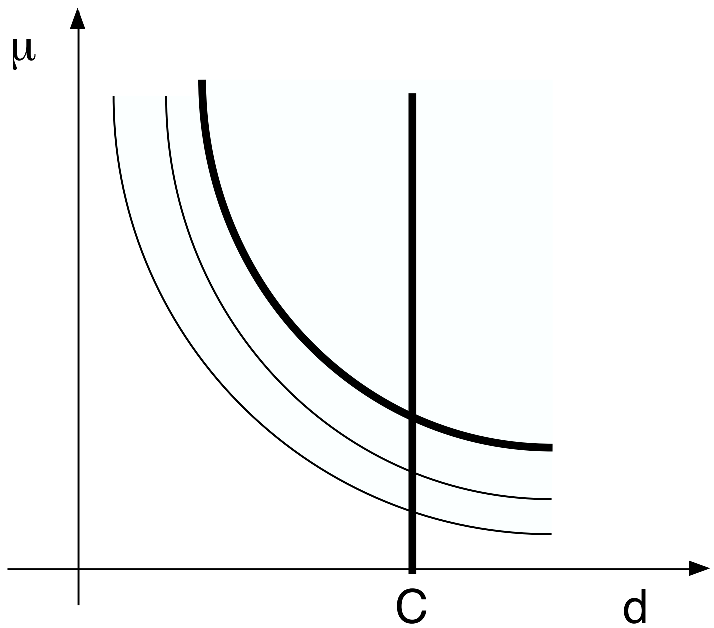
All bandwidth is allocated

Unit price of bandwidth: $\mu = \dfrac{w_1 + \cdots + w_R}{C}$

$$d_r = \frac{w_r}{\mu}$$

“Supply = Demand” Interpretation

User r submits w_r: same as submitting “demand curve”:



Mechanism clears the market:

$$\frac{w_1}{\mu} + \cdots + \frac{w_R}{\mu} = \text{total demand} = \text{supply} = C$$

Users as Price Takers

Given price μ, user r solves:

$$\max_{w_r \geq 0} \; U_r\left(\frac{w_r}{\mu}\right) - w_r$$

Theorem 1 (Existence of Competitive Equilibrium; Kelly, 1
There exist $\boldsymbol{w}$ and μ such that:

(a) w_r *is optimal for user* r *given the price* μ.

(b) $(w_1 + \cdots + w_R)/\mu = C$.

The resulting allocation is socially optimal.

Users as Price Anticipators

Suppose users *know* the price setting procedure.

Given $(w_s, s \neq r)$, user r solves:

$$\max_{w_r \geq 0} \quad U_r\left(\frac{w_r}{w_r + \sum_{s \neq r} w_s} C\right) - w_r$$

This is now a *game*, where the strategy of user r is the bid w_r

Interested in Nash equilibria

Example

$C = 1, \;\; U_1(d_1) = 2d_1, \;\; U_2(d_2) = d_2.$

Social optimum: $d_1 = 1, d_2 = 0$

Price-taking equilibria: $\mu = 1$

Price-anticipating users:

(a) $\mu > 1 \implies w_2 = 0 \implies w_1 = \;?$

(b) $\mu = 1$, $w_2 > 0$; user 2 will reduce w_2, and reduce th

(c) $\mu < 1$, $w_1 > 0$, $w_2 > 0$: inefficient

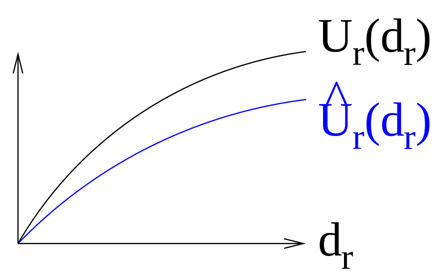
Nash Equilibrium

Theorem 2 (Hajek & Gopalakarishnan, 2002) *Assume* $R >$

There exists a unique Nash equilibrium $\boldsymbol{w}$.

The resulting allocations d_r *are the unique socially optimal so*
for modified utilities:

$$\hat{U}_r(d_r) = \left(1 - \frac{d_r}{C}\right) U_r(d_r) + \left(\frac{d_r}{C}\right) \left(\frac{1}{d_r} \int_0^{d_r} U_r(z)\, d$$



Efficiency Loss

Theorem 3 *The efficiency loss is no more than 25%:*

$$\sum_r U_r(d_r^G) \geq \frac{3}{4} \sum_r U_r(d_r^S)$$

$$\texttt{(Nash eq. utility)} \geq \frac{3}{4} \times \texttt{(socially optimal ut}$$

Furthermore, this bound is tight.

Worst case:

Many users, linear utility functions, one “dominant” user

The Worst Case

$$C = 1$$
$$U_1(d_1) = d_1$$
$$U_r(d_r) \approx d_r/2$$
$$R \to \infty$$

$$\Longrightarrow$$

$$d_1^G \to 1/2$$
$$\sum_{r>1} d_r^G \to 1/2$$
$$\mu \to 1/2$$

$$\sum_r U_r(d_r^S) = 1$$

$$\sum_r U_r(d_r^G) \to 3/4$$

Summary of Properties

1. Strategy = a “smooth” demand curve,
   chosen from a 1-parameter family ($d = w/\mu$)
2. Price is set by market clearing (supply = total demand)
3. Players’ demand is always nonnegative.

4. Price taking behavior $\Longrightarrow$ full efficiency.
5. Players’ payoffs are concave when price anticipating.

Characterization theorem:
Out of all mechanisms with the above properties,
the one we have studied minimizes the worst-case efficiency l

PART II: Networks

Link j has capacity C_j
Each user is identified with a path

Social optimum:

$$\begin{aligned} \text{maximize} \quad & \sum_r U_r(d_r) \\ \text{subject to} \quad & \text{capacity constraints} \end{aligned}$$

The Pricing Mechanism

User r submits a bid w_r^j, at each link j

Receives bandwidth at that link: $x_r^j(\boldsymbol{w}) = \dfrac{w_r^j}{w_1^j + \cdots + w_R^j} C$

User r sends as much as possible: $d_r = \min_{j \in \text{path}} x_r^j$

User r payoff: $U_r(d_r(\boldsymbol{x}_r(\boldsymbol{w}))) - \sum_j w_r^j$ Concave!

Example

Existence of a Nash Equilibrium

Theorem 4 *With "price-anticipating" users,*
there exists a Nash equilibrium $(\boldsymbol{w})$ *for an extended game.*

NE of original game maps to a NE of the extended game

Proof idea:

– Introduce "virtual user" bidding $\epsilon > 0$ at each j.
– With $\epsilon > 0$, no discontinuities, NE exists (Rosen's theorem)
– Take the limit as $\epsilon \longrightarrow 0$
– Perturbed game allocations $x_r^j(\epsilon)$ in the limit

General Networks: Efficiency Loss

Theorem 5 *The efficiency loss is no more than 25%:*

$$\texttt{(Nash eq. utility)} \geq \frac{3}{4} \times \texttt{(socially optimal uti}$$

Proof idea:

Reduce to analyzing multiple single link games, one at each j

PART III: Elastic Capacities

Supplying f units incurs cost \$$C(f)$.

$C(f)$ strictly increasing, strictly convex, nonnegative

For user r, consuming d_r units yields utility \$$U_r(d_r)$.

$U_r(d_r)$ strictly increasing, concave, nonnegative

An efficient allocation maximizes net monetary benefit:

$$\sum_r U_r(d_r) - C\left(\sum_r d_r\right)$$

Our goal: achieve an efficient allocation.

A Pricing Mechanism

Define the *price function* $p(f) = C'(f)$.

$p(f)$ strictly increasing, convex, nonnegative

We use the following mechanism (Kelly, Maulloo, and Tan, 199

(1) User r submits a bid w_r.

(2) The mechanism chooses $f(\boldsymbol{w})$ to "clear the market":

$$\begin{array}{ccccc} \sum_r w_r & = & f(\boldsymbol{w}) & \times & p(f(\boldsymbol{w})) \\ \text{Revenue} & & \text{Quantity} & \times & \text{Price} \end{array}$$

User r's rate: $d_r = w_r / p(f(\boldsymbol{w}))$

User r's payoff: $U_r(d_r) - w_r$

Price Taking Users

Suppose users assume price $p(f(\boldsymbol{w}))$ does not depend on th

Theorem 6 (Kelly, Maulloo, and Tan, 1998)

There exist $\boldsymbol{w}$ *and* μ *such that:*

1. w_r *is optimal for user* r *given the price* μ*.*
2. $\mu = p(f(\boldsymbol{w}))$*.*

Further, $\boldsymbol{d} = \boldsymbol{w}/\mu$ *is an efficient allocation.*

Price Anticipating Users

Suppose users anticipate the effect of their bid on $p(f(\boldsymbol{w}))$.

Theorem 7

There exists a Nash equilibrium $\boldsymbol{w}$, and it is unique if p is diffe

Proof idea:

Show that d_r is concave in w_r, for each user r.
Apply Rosen's existence theorem.

Price Anticipating Users: Efficiency Los

Theorem 8

The efficiency loss is no more than $\approx$ 34%:

$$\begin{matrix}\textit{Net monetary benefit} \\ \textit{at Nash equilibrium}\end{matrix} \geq (4\sqrt{2} - 5) \times \begin{matrix}\textit{Net monetar} \\ \textit{at efficient a}\end{matrix}$$

Furthermore, this bound is tight.

Worst case:

Many users, linear utility functions, piecewise linear price func

Additional Results

- If $C(f) = f^B$, then as $B \to \infty$, efficiency loss $\to$ 25%. (Intuition: *as if* the resource is in inelastic supply.)

- Results generalize to network context.

Back to Congestion Control

Faster time scale

Fixed bids w: network protocol allocates rates

Slower time scale

User observes link prices and adjusts bids

Implicit in our model:

(a) Observe individual link prices

(b) Can anticipate the effect of a bid change on prices

(c) Do not anticipate changes (reactions) in bids of other user

Interesting Directions

Dynamics of convergence to equilibrium

Non-network contexts
– auctions for divisible goods
– games between suppliers (e.g., electric power)

“Complexity” versus efficiency guarantees

Different information models